

# CHARACTER FORMULÆ AND GKRS MULTIPLETS IN EQUIVARIANT K-THEORY

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**ABSTRACT.** Let  $G$  be a compact Lie group,  $H$  a closed subgroup of maximal rank and  $X$  a topological  $G$ -space. We obtain a variety of results concerning the structure of the  $H$ -equivariant K-ring  $K_H^*(X)$  viewed as a module over the  $G$ -equivariant K-ring  $K_G^*(X)$ . One result is that the module has a nonsingular bilinear pairing; another is that the module contains multiplets which are analogous to the Gross-Kostant-Ramond-Sternberg multiplets of representation theory. We also obtain a Borel-Weil-Bott theorem on the induction of irreducible representations of  $H$ .

## INTRODUCTION

Frobenius [12, Band III, pp. 82-103] showed how to “extend” a character of a subgroup to a character of the ambient group. He considered only finite groups, but his method, which we will refer to as *formal induction*, was soon adapted in various ways to infinite and topological groups, and to this day is one of the most important technical tools of representation theory.

Induction methods for a compact Lie group  $G$  and a closed subgroup  $H$  were systematically studied by Bott [7] as an application of the index theory of equivariant elliptic operators. These index theory methods work well only if the subgroup  $H$  is of maximal rank. The purpose of this paper is to make some applications of Bott’s work and later work building on it to the equivariant K-theory of topological  $G$ -spaces. Our main results, all valid under the assumption that  $H$  is of maximal rank, are a relative duality theorem and a multiplet theorem. We also obtain a number of results on representation rings, one of which is a “Borel-Weil-Bott” theorem. Many of our results are known in important special cases, which explains the largely expository nature of this paper.

The relative duality theorem, Theorem 4.1.4, states that for every compact  $G$ -space  $X$  the  $H$ -equivariant K-group  $K_H^*(X)$  is equipped with a  $K_G^*(X)$ -bilinear nonsingular pairing. This result contains as a special case a form of Poincaré duality for the K-group of the homogeneous space  $G/H$ , and it generalizes results of Pittie [27], Steinberg [36], Shapiro [32], [33], McLeod [26], and Kazhdan and Lusztig [18]. For the special case of a maximal torus this result can be found in our earlier paper [15].

The multiplet theorem, Theorem 4.2.2, generalizes a result of Gross et al. [13], which says that the irreducible representations of  $H$  are naturally partitioned into multiplets which have the property that the alternating sum of the dimensions of the modules in each multiplet is zero. Our theorem states that the  $H$ -equivariant

K-group  $K_H^*(X)$  of a compact  $G$ -space  $X$  is similarly divided into multiplets. Under the forgetful map the classes in each multiplet map to classes in the ordinary K-group  $K^*(X)$  with the property that the alternating sum is zero.

We have included some elementary background material. For instance, because the homogeneous space  $G/H$  is not always K-orientable (i.e. does not always have a  $\mathbf{Spin}^c$ -structure), it is necessary to incorporate an orientation twist in the statement of our theorems, and in fact also in the definition of the induction map. This necessity was overlooked by Bott, which led to a number of minor errors in his paper. In §1 we offer a review of Bott's work, in which we take care to correct these mistakes. In Appendix C we classify  $\mathbf{Spin}^c$ -structures on  $G/H$ , which is an easy exercise, but which to our surprise we could not find in the literature. In a separate paper [25] we will list examples where such structures do not exist (the simplest of which is the Grassmannian of oriented 3-planes in  $\mathbf{R}^7$ ).

Much of the material in this paper grew out of discussions with Megumi Harada. The authors are grateful for her help and encouragement. We also thank Eckhard Meinrenken for several helpful suggestions.

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## 1. INDUCTION MAPS

This section is a review of Bott's paper [7] on equivariant index theory for compact homogeneous spaces, where we extend his work by allowing for certain twists of representation rings. This amounts to working not with ordinary representations but with projective representations with a suitable central character. Although this is a routine generalization, the particulars are confusing and not always correctly recorded in the literature, which is why we provide a detailed treatment. The results are most conveniently formulated in terms of twisted K-theory in the sense of Donovan and Karoubi [11]. (We will only need to twist K-theory by torsion classes; we do not require Rosenberg's more general version of this theory.) An induction map is then a wrong-way or pushforward homomorphism between equivariant twisted K-groups. In ordinary K-theory pushforward maps are commonly defined by means of a Dirac operator; in twisted K-theory we employ what we call a twisted Dirac operator. This term usually means a Dirac operator with coefficients in a bundle, but in this paper we will use it for a special type of transversely elliptic operator in the sense of Atiyah [2], which lives not on the base manifold but on a suitable principal bundle over it.

*Notation.* Our global notational conventions are as follows. See also the index at the end.

Throughout this paper  $G$  denotes a compact connected Lie group,  $\mathcal{X}(G) = \text{Hom}(G, \mathbf{U}(1))$  the character group of  $G$ , and  $R(G)$  the Grothendieck ring of finite-dimensional complex  $G$ -modules. A character  $\chi \in \mathcal{X}(G)$  determines a one-dimensional  $G$ -module  $\mathbf{C}_\chi$  and hence a class in  $R(G)$ , which we denote by  $e^\chi$ .

We choose once and for all a maximal torus  $T$  of  $G$ . We denote the inclusion map by  $j_G: T \rightarrow G$  and the Weyl group by  $W_G = N_G(T)/T$ . The map  $\mathcal{X}(T) \rightarrow R(T)$  defined by  $\chi \mapsto e^\chi$  identifies  $R(T)$  with the group ring  $\mathbf{Z}[\mathcal{X}(T)]$ . The restriction homomorphism  $j_G^*: R(G) \rightarrow R(T)$  induces an isomorphism  $R(G) \cong R(T)^{W_G}$ . (See e.g. [8, §IX.3].) In particular the ring  $R(G)$  has no zero divisors. Let  $\mathcal{R}_G \subseteq \mathcal{X}(T)$  be the root system of  $(G, T)$ . We fix a basis  $\mathcal{B}_G$  of  $\mathcal{R}_G$ , which determines a set of positive roots  $\mathcal{R}_G^+$ , and we let

$$\rho_G = \frac{1}{2} \sum_{\alpha \in \mathcal{R}_G^+} \alpha \in \frac{1}{2} \mathcal{X}(T)$$

be the half-sum of the positive roots.

We denote by  $H$  a closed subgroup of  $G$  with inclusion map  $i: H \rightarrow G$ . From §1.5 on we will assume that  $H$  is connected and contains  $T$ . We let  $M$  be the homogeneous space  $G/H$  and  $\bar{1} = 1H$  the identity coset. For any  $H$ -module  $V$ , we denote by

$$G \times^H V$$

the homogeneous vector bundle on  $M$  with fibre at  $\bar{1}$  equal to  $V$ . This notation is not to be confused with

$$A \times_C B,$$

which we use for the fibred product of two objects  $A$  and  $B$  in some category (e.g. the category of groups) over a third object  $C$ . The tangent space to  $M$  at  $\bar{1}$  is denoted by  $\mathfrak{m}$  and the isotropy representation by

$$\eta: H \longrightarrow \mathbf{GL}(\mathfrak{m}).$$

The  $H$ -module  $\mathfrak{m}$  is naturally isomorphic to  $\mathfrak{g}/\mathfrak{h}$ ; and the tangent bundle  $TM$  is naturally isomorphic to  $G \times^H \mathfrak{m}$ . We choose a  $G$ -invariant inner product on  $\mathfrak{g}$ , and we identify  $\mathfrak{g}$  with  $\mathfrak{g}^*$  and  $\mathfrak{m}$  with the orthogonal complement of  $\mathfrak{h}$  in  $\mathfrak{g}$ .

**1.1. Formal induction.** Let  $R(G)^* = \text{Hom}_{\mathbf{Z}}(R(G), \mathbf{Z})$  be the dual abelian group of  $R(G)$ . As in [7, §1] we will identify  $R(G)^*$  with the Grothendieck group of locally finite  $G$ -modules, i.e. the group of possibly infinite  $\mathbf{Z}$ -linear combinations  $\sum_k a_k [E_k]$  of isomorphism classes of irreducible  $G$ -modules  $E_k$ . If  $E$  is a locally finite  $G$ -module and  $F$  a finite  $G$ -module, then the dual pairing between the classes  $[E] \in R(G)^*$  and  $[F] \in R(G)$  is given by

$$\langle [E], [F] \rangle = \dim \text{Hom}_G(F, E). \quad (1.1)$$

The *formal pushforward* or *formal induced module*  $i_!(V) = \text{ind}_H^G(V)$  of a finite-dimensional  $H$ -module  $V$  is the locally finite part of the  $G$ -module of smooth global sections  $\Gamma(M, G \times^H V)$ . The *formal pushforward homomorphism* or *formal induction map* is the  $\mathbf{Z}$ -linear map

$$i_!: R(H) \rightarrow R(G)^*$$

so defined. It is  $R(G)$ -linear in the sense that  $i_!(i^*(b)\phi) = bi_!(\phi)$  for all  $b \in R(G)$  and  $\phi \in R(H)^*$ . Formal induction satisfies Frobenius reciprocity, which can be stated by saying that  $i_!$  is the composition of the two maps

$$i_! : R(H) \longrightarrow R(H)^* \xrightarrow{i^*} R(G)^*.$$

Here  $R(H) \rightarrow R(H)^*$  is the natural inclusion and  $i^*$  is the transpose of  $i_!$ . (See [7, § 2] or [30, § 2].) Frobenius reciprocity means that formal induction is the adjoint of the restriction functor, which implies that

$$i_!(V) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V \quad (1.2)$$

as a  $\mathfrak{g}$ -module, where  $\mathfrak{g}$  acts by left multiplication on the universal enveloping algebra  $U(\mathfrak{g})$ . (See [6].)

A less desirable property is that the module  $i_!(V)$  is often infinite-dimensional even if  $V$  is finite-dimensional. We wish to look for induction maps that preserve finite-dimensionality, which forces us to give up Frobenius reciprocity. Accordingly, by an *induction map* we will mean any  $R(G)$ -linear map  $R(H) \rightarrow R(G)$ , i.e. any element of

$$R(H)^\vee = \text{Hom}_{R(G)}(R(H), R(G)),$$

the dual of the  $R(G)$ -module  $R(H)$ .

**1.2. Twisted induction.** We require a slightly more general notion involving projective representations. Part of the following material is taken from [24]. Let

$$\sigma : 1 \longrightarrow \mathbf{U}(1) \longrightarrow G^{(\sigma)} \longrightarrow G \longrightarrow 1 \quad (1.3)$$

be a central extension of  $G$  by the circle  $\mathbf{U}(1)$ . (A notational convention: the label  $\sigma$  will refer to the exact sequence (1.3) as a whole, but when there is no danger of confusion, we will speak of “the extension  $G^{(\sigma)}$ .”) A complex  $G^{(\sigma)}$ -module  $V$  has *central character* or *level*  $k \in \mathbf{Z}$  if the subgroup  $\mathbf{U}(1)$  acts on  $V$  by  $z \cdot v = z^k v$ . Let  $\mathfrak{Rep}(G, \sigma)$  be the category of finite-dimensional complex  $G^{(\sigma)}$ -modules of level 1. We call the Grothendieck group  $R(G, \sigma)$  of  $\mathfrak{Rep}(G, \sigma)$  the  *$\sigma$ -twisted representation module* of  $G$ . In the case of the trivial extension  $\sigma = 0$  (which is defined by  $G^{(0)} = \mathbf{U}(1) \times G$ ) there is an evident equivalence of categories  $\mathfrak{Rep}(G, 0) \rightarrow \mathfrak{Rep}(G)$ , which identifies  $R(G, 0)$  with  $R(G)$ .

Recall that the *sum* of two central extensions  $\sigma$  and  $v$  of  $G$  by  $\mathbf{U}(1)$  is the central extension  $\sigma + v$  of  $G$  by  $\mathbf{U}(1)$  defined by

$$G^{(\sigma+v)} = \hat{G}/K.$$

Here  $\hat{G}$  is the fibred product  $G^{(\sigma)} \times_G G^{(v)}$  and  $K$  is a copy of  $\mathbf{U}(1)$  anti-diagonally embedded in  $\hat{G}$ . The extension *opposite* to  $\sigma$  is the extension  $-\sigma$  obtained by precomposing the inclusion  $\mathbf{U}(1) \rightarrow G^{(\sigma)}$  with the automorphism  $z \mapsto z^{-1}$  of  $\mathbf{U}(1)$ . The tensor product functor

$$\mathfrak{Rep}(G, \sigma) \times \mathfrak{Rep}(G, v) \longrightarrow \mathfrak{Rep}(G, \sigma + v)$$

induces a bi-additive map

$$R(G, \sigma) \times R(G, v) \longrightarrow R(G, \sigma + v). \quad (1.4)$$

By taking  $\sigma = 0$ , we see that  $R(G, v)$  is an  $R(G)$ -module for all  $v$ . With respect to this module structure, the map (1.4) is  $R(G)$ -bilinear.

**1.2.1. Remark.** If two extensions  $\sigma$  and  $v$  are equivalent, then the twisted representation modules  $R(G, \sigma)$  and  $R(G, v)$  are isomorphic, but the isomorphism depends on the choice of the equivalence. For this reason we do *not* identify  $R(G, \sigma)$  with  $R(G, v)$  unless an explicit equivalence  $\sigma \sim v$  has been specified.

For any extension  $G^{(\sigma)}$  as in (1.3) we denote by  $H^{(\sigma)}$  the preimage of  $H$  under the projection  $G^{(\sigma)} \rightarrow G$ . The group  $H^{(\sigma)}$  is a central extension of  $H$  by  $\mathbf{U}(1)$  (namely the pullback of the extension  $G^{(\sigma)}$  via the inclusion map  $i: H \rightarrow G$ ). The map  $G^{(\sigma)} \rightarrow G$  descends to a diffeomorphism  $G^{(\sigma)}/H^{(\sigma)} \cong M$ , which allows us to identify  $M$  with the homogeneous space  $G^{(\sigma)}/H^{(\sigma)}$ .

**1.2.2. Lemma.** *Let  $\sigma$  and  $v$  be central extensions of  $G$  by  $\mathbf{U}(1)$ .*

- (i) *The  $R(G)$ -module  $R(G^{(\sigma)})$  is the direct sum of the submodules  $R(G, k\sigma)$  over all levels  $k \in \mathbf{Z}$ . Each summand  $R(G, k\sigma)$  is nonzero.*
- (ii) *The group  $T^{(\sigma)}$  is a maximal torus of  $G^{(\sigma)}$ ; the submodule  $R(T, \sigma)$  of  $R(T^{(\sigma)})$  is preserved by the  $W_G$ -action; and the restriction homomorphism  $R(G, \sigma) \rightarrow R(T, \sigma)$  is an isomorphism onto  $R(T, \sigma)^{W_G}$ .*
- (iii) *Let  $a \in R(G, \sigma)$  and  $b \in R(G, v)$ . If  $ab \in R(G, \sigma + v)$  is equal to 0, then  $a = 0$  or  $b = 0$ .*

*Proof.* (i) and (ii) are special cases of Lemma A.1 in the Appendix (where  $G^{(\sigma)}$  is denoted by  $\check{G}$  and  $R(G, k\sigma)$  by  $R^k(\check{G})$ ). The group  $\hat{G} = G^{(\sigma)} \times_G G^{(v)}$  is a central  $\mathbf{U}(1)$ -extension of  $G^{(\sigma)}$  as well as of  $G^{(v)}$ . Therefore, by (i), the modules  $R(G, \sigma)$  and  $R(G, v)$  are naturally isomorphic to submodules of  $R(\hat{G})$ . Under this isomorphism, the bilinear map (1.4) corresponds to multiplication in  $R(\hat{G})$ . Since  $\hat{G}$  is a central extension of  $G$  by the torus  $\mathbf{U}(1)^2$ , it is connected, and therefore  $R(\hat{G})$  has no zero divisors. QED

**1.2.3. Example.** Suppose  $G = T$  is a torus. Choose a character  $\mu$  of  $T^{(\sigma)}$  of level 1. Then the character group  $\mathcal{X}(T^{(\sigma)})$  is the direct sum of  $\mathcal{X}(T)$  and  $\mathbf{Z}\mu$ , and therefore  $R(T^{(\sigma)}) = R(T)[e^\mu, e^{-\mu}]$  is a Laurent polynomial algebra over  $R(T)$  in one variable. The level  $k$  submodule is  $R(T, k\sigma) = R(T)e^{k\mu}$ , which is a free  $R(T)$ -module of rank 1 on the generator  $e^{k\mu}$ .

**1.2.4. Lemma.** *Let  $\sigma$  be a central extension of  $G$  by  $\mathbf{U}(1)$  and let  $V$  be an  $H^{(\sigma)}$ -module of level 1. Then the space of smooth sections of the  $G^{(\sigma)}$ -homogeneous vector bundle  $G^{(\sigma)} \times^{H^{(\sigma)}} V$  over  $M$  is a  $G^{(\sigma)}$ -module of level 1.*

*Proof.* Put  $E = G^{(\sigma)} \times^{H^{(\sigma)}} V$ . For  $g \in G^{(\sigma)}$  and  $x \in V$ , let us write  $[g]$  for the coset  $gH^{(\sigma)} \in M$  and  $[g, x]$  for the element  $H^{(\sigma)} \cdot (g, x) \in E$ . A section  $s \in \Gamma(M, E)$  can be presented in the form

$$s([g]) = [g, f(g)],$$

where  $f: G^{(\sigma)} \rightarrow V$  is a smooth function satisfying  $f(gh^{-1}) = h \cdot f(g)$  for all  $h \in H^{(\sigma)}$ . An element  $k \in G^{(\sigma)}$  acts on  $s$  by

$$(k \cdot s)([g]) = k \cdot (s(k^{-1} \cdot [g])) = k \cdot [k^{-1}g, f(k^{-1}g)] = [g, f(k^{-1}g)].$$

In particular, since  $V$  is an  $H^{(\sigma)}$ -module of level 1, an element  $z \in \mathbf{U}(1) \subseteq H^{(\sigma)} \subseteq G^{(\sigma)}$  acts by

$$(z \cdot s)([g]) = [g, f(z^{-1}g)] = [g, z \cdot f(g)] = [g, zf(g)] = z[g, f(g)] = zs([g]),$$

i.e.  $z \cdot s = zs$ .

QED

**1.2.5. Definition.** A *twisted induction map* is an element of

$$\mathrm{Hom}_{R(G)}(R(H, \tau), R(G, \sigma)),$$

where  $\sigma$  is a central extension of  $G$  by  $\mathbf{U}(1)$  and  $\tau$  is a central extension of  $H$  by  $\mathbf{U}(1)$ .

**1.3. Elliptic operators.** We will obtain twisted induction maps from transversely elliptic differential operators. To motivate Definition 1.3.1 below let us first consider the untwisted case. Consider a compact  $G \times H$ -manifold  $X$  and a  $G \times H$ -equivariant differential operator  $D$  on  $X$  which is transversely elliptic with respect to  $H$ . There are a number of equivalent ways to define the *equivariant index* of  $D$ . We will define it as the locally finite part of  $\ker(D)$  minus the locally finite part of  $\ker(D^*)$ . By the results of [2, Lecture 2] the index is an element of

$$\mathrm{index}(D) \in R(H)^* \otimes_{\mathbf{Z}} R(G), \quad (1.5)$$

and so can be viewed as a  $\mathbf{Z}$ -linear map from  $R(H)$  to  $R(G)$ . This map is in fact  $R(G)$ -linear and is therefore an induction map in our sense. To get twisted induction maps we will replace  $G$  and  $H$  by central extensions  $G^{(\sigma)}$  and  $H^{(\tau)}$ . We will take  $D$  to be an operator on  $X = G^{(\sigma)}$  with coefficients in a  $\mathbf{Z}/2\mathbf{Z}$ -graded  $H^{(\tau)}$ -module of level 1. Even though  $D$  does not descend to an operator on the homogeneous space  $M = G^{(\sigma)}/H^{(\sigma)}$  (except when  $\tau = 0$ ), we will call it a “twisted” operator on  $M$ .

Let  $\sigma$  and  $\tau$  be as in Definition 1.2.5 and put

$$\hat{H} = H^{(\sigma)} \times_H H^{(\tau)}.$$

For any  $H^{(\tau)}$ -module  $U$  of level 1, the product  $G^{(\sigma)} \times U$  is a  $G^{(\sigma)} \times \hat{H}$ -equivariant vector bundle over  $G^{(\sigma)}$ . Here  $G^{(\sigma)}$  acts on itself by left multiplication, and  $\hat{H}$  acts on  $G^{(\sigma)}$  by right multiplication via the homomorphism  $\hat{H} \rightarrow H^{(\sigma)}$  and on  $U$  via the homomorphism  $\hat{H} \rightarrow H^{(\tau)}$ .

**1.3.1. Definition.** Let  $U^0$  and  $U^1$  be  $H^{(\tau)}$ -modules of level 1. A  $(\sigma, \tau)$ -*twisted equivariant elliptic differential operator* on  $M$  (or a  $(\sigma, \tau)$ -*twisted operator* for short) is a  $G^{(\sigma)} \times \hat{H}$ -equivariant linear differential operator

$$D: \Gamma(G^{(\sigma)}, G^{(\sigma)} \times U^0) \longrightarrow \Gamma(G^{(\sigma)}, G^{(\sigma)} \times U^1)$$

on  $G^{(\sigma)}$  which is transversely elliptic relative to  $\hat{H}$ .

If  $\tau = 0$  is the trivial extension, then a  $(\sigma, \tau)$ -twisted operator descends to a  $G^{(\sigma)}$ -equivariant elliptic differential operator on  $M$  in the usual sense.

Following (1.5) we view the index of a  $(\sigma, \tau)$ -twisted operator  $D$  as a  $\mathbf{Z}$ -linear map  $\mathrm{index}(D): R(\hat{H}) \rightarrow R(G^{(\sigma)})$ .

**1.3.2. Lemma.** Let  $D$  be a  $(\sigma, \tau)$ -twisted operator.

- (i) The index of  $D$  is  $R(G^{(\sigma)})$ -linear and maps  $R(H, \tau - \sigma)$  to  $R(G, \sigma)$ .

- (ii) Let  $V$  be an  $H^{(\tau-\sigma)}$ -module of level 1. For  $p = 0, 1$ , let  $E_V^p$  be the  $G^{(\sigma)}$ -homogeneous vector bundle  $G^{(\sigma)} \times^{H^{(\sigma)}} (U^p \otimes V^*)$  over  $M$ . Then  $D$  descends to a  $G^{(\sigma)}$ -equivariant elliptic differential operator

$$D_V: \Gamma(M, E_V^0) \longrightarrow \Gamma(M, E_V^1)$$

with the property that  $\text{index}(D_V) = \langle \text{index}(D), [V] \rangle$ .

*Proof.* The  $R(G^{(\sigma)})$ -linearity is a special case of the multiplicativity property of the index, [2, Theorem 3.5]. (Take  $X = \text{pt}$  and  $Y = M$  in the statement of that theorem.) Let  $V$  be an  $H^{(\tau-\sigma)}$ -module of level 1. Then for  $p = 0, 1$  the tensor product  $U^p \otimes V^*$  is an  $H^{(\sigma)}$ -module of level 1, and the bundle  $G^{(\sigma)} \times (U^p \otimes V^*)$  over  $G^{(\sigma)}$  is  $G^{(\sigma)} \times H^{(\sigma)}$ -equivariant. This shows that the vector bundle  $E_V^p$  is well-defined. It follows from (1.1) that

$$\langle \text{index}(D), [V] \rangle = [\text{Hom}_{\hat{H}}(V, \ker(D))] - [\text{Hom}_{\hat{H}}(V, \ker(D^*))] \in R(G^{(\sigma)}). \quad (1.6)$$

Now  $\text{Hom}_{\hat{H}}(V, \ker(D)) \cong (V^* \otimes \ker(D))^{\hat{H}}$  is isomorphic to a  $G^{(\sigma)}$ -submodule of

$$(V^* \otimes \Gamma(G^{(\sigma)}, G^{(\sigma)} \times U^0))^{\hat{H}} \cong \Gamma(G^{(\sigma)}, G^{(\sigma)} \times (U^0 \otimes V^*))^{\hat{H}} \cong \Gamma(M, E^0), \quad (1.7)$$

and is therefore of level 1 by Lemma 1.2.4. Similarly,  $\text{Hom}_{\hat{H}}(V, \ker(D^*))$  is isomorphic to a  $G^{(\sigma)}$ -submodule of  $\Gamma(M, E^1)$  and so is of level 1. It now follows from (1.6) that  $\langle \text{index}(D), [V] \rangle \in R(G, \sigma)$ , which proves (i). The operator

$$\text{id}_{V^*} \otimes D: V^* \otimes \Gamma(G^{(\sigma)}, G^{(\sigma)} \times U^0) \longrightarrow V^* \otimes \Gamma(G^{(\sigma)}, G^{(\sigma)} \times U^1)$$

is  $\hat{H}$ -equivariant. Taking  $\hat{H}$ -invariants on both sides and composing with the natural isomorphism (1.7) we get an operator  $D_V: \Gamma(M, E_V^0) \longrightarrow \Gamma(M, E_V^1)$ , which is a  $G^{(\sigma)}$ -equivariant differential operator because  $D$  is. The principal symbol of  $D$  is a morphism of  $G^{(\sigma)} \times \hat{H}$ -equivariant bundles over  $T_{\hat{H}}^* G^{(\sigma)} = G^{(\sigma)} \times \mathfrak{m}$ ,

$$\text{symbol}(D): G^{(\sigma)} \times \mathfrak{m} \times U^0 \longrightarrow G^{(\sigma)} \times \mathfrak{m} \times U^1,$$

which is an isomorphism off the zero section  $G^{(\sigma)} \times \mathfrak{m} \times \{0\}$ . This transversely elliptic symbol induces an elliptic symbol

$$(G^{(\sigma)} \times \mathfrak{m}) \times^{H^{(\sigma)}} (U^0 \otimes V^*) \longrightarrow (G^{(\sigma)} \times \mathfrak{m}) \times^{H^{(\sigma)}} (U^1 \otimes V^*)$$

on  $T^*M = (G^{(\sigma)} \times \mathfrak{m})/H^{(\sigma)}$ , which is the symbol of  $D_V$ . Hence  $D_V$  is elliptic. If  $V$  is irreducible, then under the isomorphism (1.7) the kernel of  $D_V$  corresponds to the  $V$ -isotypical subspace of  $\ker(D)$ . Similarly,  $\ker(D_V^*)$  corresponds to the  $V$ -isotypical subspace of  $\ker(D^*)$ . Therefore, by (1.6),  $\text{index}(D_V) = \langle \text{index}(D), [V] \rangle$  for irreducible  $V$ . The index of  $D_V$  is additive with respect to  $V$ , so we conclude that  $\text{index}(D_V) = \langle \text{index}(D), [V] \rangle$  for general  $V$ . QED

**1.3.3. Definition.** Let  $D$  be a  $(\sigma, \tau)$ -twisted operator. The *associated induction map* is the map

$$i_D \in \text{Hom}_{R(G)}(R(H, \sigma - \tau), R(G, \sigma))$$

defined by  $i_D([V]) = \text{index}(D_{V^*}) = \langle \text{index}(D), [V^*] \rangle$ .

Let  $\zeta: M \rightarrow T^*M$  be the zero section of the cotangent bundle and let  $T^\times M = T^*M \setminus \zeta(M)$  be the punctured cotangent bundle. The class of the principal symbol of a  $(\sigma, \tau)$ -twisted operator  $D$  is an element of the relative twisted K-group,

$$\mathbf{th}(D) = [\text{symbol}(D)] \in K_G^0(T^*M, T^\times M, \tau), \quad (1.8)$$

called the *equivariant Thom class* of  $D$ . Its restriction to the zero section,

$$\mathbf{e}(D) = \zeta^*(\mathbf{th}(D)) = [U^0] - [U^1] \in K_G^0(M, \tau) \cong K_H^0(\text{pt}, \tau) \cong R(H, \tau), \quad (1.9)$$

is the *equivariant Euler class* of  $D$ . (The extension  $\tau$  is characterized by a cohomology class  $[\tau] \in H_H^3(\text{pt}, \mathbf{Z}) \cong H_G^3(M, \mathbf{Z})$  and so can be viewed as an  $H$ -equivariant twist of a point or a  $G$ -equivariant twist of  $M$ . By definition the twisted K-group  $K_G^0(M, \tau)$  is then the Grothendieck group of  $G \times H^{(\tau)}$ -equivariant vector bundles on  $G$  which are of level 1 with respect to the  $H^{(\tau)}$ -action. For any  $H$ -space  $X$  the twisted K-group  $K_H^0(X, \tau)$  is the Grothendieck group of  $H^{(\tau)}$ -equivariant vector bundles on  $X$  of level 1. See e.g. [9] on how to reconcile this definition of twisted K-theory with that of [11].)

The following statement, which summarizes and extends to the twisted case results of Bott [7], says that an induction map defined by a twisted operator depends only on the Euler class. Moreover, all such induction maps vanish if the subgroup is not of maximal rank. The situation in the maximal rank case is diametrically opposite and is described in Theorem 3.4.3 below. For simplicity we denote the inclusion  $H^{(\sigma)} \rightarrow G^{(\sigma)}$  by  $i$  and the formal induction map  $R(H^{(\sigma)}) \rightarrow R(G^{(\sigma)})^*$  by  $i_!$ . It follows from Lemma 1.2.4 that  $i_!$  maps  $R(H, \sigma)$  to  $R(G, \sigma)^*$ .

**1.3.4. Theorem.** *Let  $D$  be a  $(\sigma, \tau)$ -twisted operator on  $M$  and let  $i_D: R(H, \sigma - \tau) \rightarrow R(G, \sigma)$  be the associated induction map.*

- (i)  $i_D(a) = i_!(\mathbf{e}(D)a)$  for all  $a \in R(H, \sigma - \tau)$ .
- (ii)  $i_D = 0$  if  $H$  does not contain a maximal torus of  $G$ .
- (iii) If  $\tau = 0$ , then  $i_D(1) = i_!(\mathbf{e}(D))$  and  $i_D i^*(b) = i_D(1)b$  for all  $b \in R(G, \sigma)$ .

*Proof.* Let  $V$  be an  $H^{(\sigma-\tau)}$ -module of level 1. The Euler class of the operator  $D_{V^*}$  defined in Lemma 1.3.2(ii) is

$$\mathbf{e}(D_{V^*}) = [U^0 \otimes V] - [U^1 \otimes V] = \mathbf{e}(D) \cdot [V] \in R(H, \sigma)$$

By [7, Theorem I], the element  $i_!(\mathbf{e}(D_{V^*})) \in R(G, \sigma)^*$  is in  $R(G, \sigma)$ , and the index of  $D_{V^*}$  is equal to  $i_!(\mathbf{e}(D_{V^*}))$ . Therefore, by Lemma 1.3.2(ii),

$$i_D([V]) = \text{index}(D_{V^*}) = i_!(\mathbf{e}(D) \cdot [V]),$$

which proves (i). (ii) follows from [7, Theorem II] if  $H$  is connected. In the general case we argue as in [30, §2]: the character of the virtual  $G^{(\sigma)}$ -module  $i_D(a)$  is given by the Lefschetz fixed point formula of Atiyah and Bott [3], but if  $H$  does not contain a maximal torus of  $G$ , then  $H^{(\sigma)}$  does not contain a maximal torus of  $G^{(\sigma)}$ , so  $T^{(\sigma)}$  acts on  $M$  without fixed points, so  $i_D(a) = 0$ . Finally, (iii) follows immediately from (i) and the  $R(G)$ -linearity of  $i_D$  (Lemma 1.3.2(i)). QED

**1.4. Twisted Dirac operators.** The operators of most interest to us will be twisted versions of Atiyah and Singer's generalized Dirac operators. We state the definition here and we will compute the Euler class of three special twisted Dirac operators in Example 1.5.2, Lemma 2.2.1(iv), and Proposition C.4.



Let  $\sigma$  be a central extension of  $G$  by  $U(1)$  and  $\tau$  a central extension of  $H$  by  $U(1)$ . Let  $\mathbf{Cl}(\mathfrak{m}_{\mathbb{C}})$  be the Clifford algebra of  $\mathfrak{m}_{\mathbb{C}}$  with respect to the complex bilinear extension of the inner product on  $\mathfrak{m}$ . The orthogonal representation  $\eta: H \rightarrow \mathbf{O}(\mathfrak{m})$  extends uniquely to a representation of  $H$  on the algebra  $\mathbf{Cl}(\mathfrak{m}_{\mathbb{C}})$  and hence to a representation of  $H^{(\tau)}$  on  $\mathbf{Cl}(\mathfrak{m}_{\mathbb{C}})$ . Let  $U$  be a  $\mathbb{Z}/2\mathbb{Z}$ -graded  $H^{(\tau)}$ -equivariant  $\mathbf{Cl}(\mathfrak{m}_{\mathbb{C}})$ -module, i.e. a  $\mathbb{Z}/2\mathbb{Z}$ -graded  $\mathbf{Cl}(\mathfrak{m}_{\mathbb{C}})$ -module  $U = U^0 \oplus U^1$  which is at the same time a  $\mathbb{Z}/2\mathbb{Z}$ -graded  $H^{(\tau)}$ -module with the property that

$$h \cdot (cu) = (h \cdot c)(h \cdot u)$$

for all  $h \in H^{(\tau)}$ ,  $c \in \mathbf{Cl}(\mathfrak{m}_{\mathbb{C}})$  and  $u \in U$ . Assume in addition that the  $H^{(\tau)}$ -action on  $U$  is of level 1.

Let  $\theta \in \Omega(G^{(\sigma)}, \mathfrak{h}^{(\sigma)})$  be a  $G^{(\sigma)}$ -invariant connection 1-form on the principal  $H^{(\sigma)}$ -bundle  $G^{(\sigma)} \rightarrow M$ , for instance the one defined by the orthogonal splitting  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ . The trivial vector bundle  $G^{(\sigma)} \times U^p$  has a linear connection  $\nabla = d + \theta$  defined by

$$\nabla_v f = df(v) + \theta(v) \cdot f$$

for all tangent vectors  $v$  to  $G^{(\sigma)}$  and smooth functions  $f: G^{(\sigma)} \rightarrow U$ . Here we let  $\theta(v) \in \mathfrak{h}^{(\sigma)}$  act on  $f$  by combining the infinitesimal right multiplication action on  $G^{(\sigma)}$  with the action

$$\mathfrak{h}^{(\sigma)} \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{o}(\mathfrak{m}_{\mathbb{C}}) \longrightarrow \mathbf{cl}(\mathfrak{m}_{\mathbb{C}}) \longrightarrow \mathfrak{gl}(U)$$

on  $U$ . The connection  $\nabla$  is  $G^{(\sigma)} \times \hat{H}$ -equivariant, where  $\hat{H} = H^{(\sigma)} \times_H H^{(\tau)}$ . We let  $p: \mathfrak{g}^{(\sigma)} \rightarrow \mathfrak{m}$  be the canonical projection and  $\text{cliff}: \mathfrak{m} \times U^0 \rightarrow U^1$  the Clifford multiplication, and we trivialize the tangent bundle  $TG^{(\sigma)}$  via left-translations. We form a first-order operator on  $G^{(\sigma)}$ ,

$$\begin{aligned} D: \Gamma(G^{(\sigma)}, G^{(\sigma)} \times U^0) &\xrightarrow{\nabla} \Gamma(G^{(\sigma)}, G^{(\sigma)} \times \mathfrak{g}^{(\sigma)} \times U^0) \\ &\xrightarrow{p} \Gamma(G^{(\sigma)}, G^{(\sigma)} \times \mathfrak{m} \times U^0) \xrightarrow{\text{cliff}} \Gamma(G^{(\sigma)}, G^{(\sigma)} \times U^1). \end{aligned}$$

This is a  $(\sigma, \tau)$ -twisted equivariant elliptic differential operator in the sense of Definition 1.3.1, which we refer to as the *twisted Dirac operator* determined by the equivariant Clifford module  $U$  and the connection  $\theta$ . The symbol of  $D$ , and therefore the induction map  $i_D$ , are independent of the choice of  $\theta$ , which is why we will sometimes allow ourselves to speak of “the” operator determined by  $U$  without specifying  $\theta$ .

**1.5. The maximal rank case.** In view of Theorem 1.3.4(ii) we assume for the remainder of the paper that the subgroup  $H$  of  $G$  contains  $T$ . For simplicity we will also assume  $H$  to be connected. Then the homogeneous space  $M = G/H$  is simply connected ([8, Ch. IX, §2.4]). In this section we will calculate the pushforward homomorphism defined by a twisted elliptic operator on  $M$  by means of the Lefschetz formula of Atiyah and Bott.

*Notation.* We denote the various inclusion maps by

$$T \xrightarrow{j_G} G, \quad T \xrightarrow{j_H} H \xrightarrow{i} G.$$

We let  $G^{(\sigma)}$  be a central  $\mathbf{U}(1)$ -extension of  $G$  and  $H^{(\tau)}$  a central  $\mathbf{U}(1)$ -extension of  $H$ . Then we have a pullback extension  $H^{(\sigma)}$  of  $H$  and two pullback extensions  $T^{(\sigma)}$  and  $T^{(\tau)}$  of  $T$ , and corresponding inclusion maps

$$T^{(\sigma)} \xrightarrow{j_G} G^{(\sigma)}, \quad T^{(\sigma)} \xrightarrow{j_H} H^{(\sigma)}, \quad T^{(\tau)} \xrightarrow{j_H} H^{(\tau)}, \quad H^{(\sigma)} \xrightarrow{i} G^{(\sigma)}.$$

The root system of  $G$  contains that of  $H$ . We let  $\mathcal{B}_H$  be the unique basis of  $\mathcal{R}_H$  such that the associated positive roots  $\mathcal{R}_H^+$  are positive for  $G$ . We define

$$W^H = \{ w \in W_G \mid w(\mathcal{R}_H^+) \subseteq \mathcal{R}_G^+ \}.$$

Then  $W^H$  is a system of coset representatives for  $W_G/W_H$ . The set  $\mathcal{R}_M = \mathcal{R}_G \setminus \mathcal{R}_H$  (which is usually not a root system) is the set of weights of the complexified  $T$ -module  $\mathfrak{m}_{\mathbf{C}}$ . We call the weights in  $\mathcal{R}_M^+ = \mathcal{R}_G^+ \setminus \mathcal{R}_H^+$  *positive*; the set  $\mathcal{R}_M$  is the disjoint union of  $\mathcal{R}_M^+$  and  $-\mathcal{R}_M^+$ . The dimension of  $M$  is the cardinality of  $\mathcal{R}_M$ , which is even. For any root  $\alpha$  of  $G$ , let  $\mathfrak{g}_{\mathbf{C}}^\alpha \subseteq \mathfrak{g}_{\mathbf{C}}$  be the root space corresponding to  $\alpha$ . We have

$$\mathfrak{m}_{\mathbf{C}} = \bigoplus_{\alpha \in \mathcal{R}_M} \mathfrak{g}_{\mathbf{C}}^\alpha, \quad \mathfrak{m} = \bigoplus_{\alpha \in \mathcal{R}_M^+} \mathfrak{m}^\alpha,$$

where  $\mathfrak{m}^\alpha = \mathfrak{m} \cap (\mathfrak{g}_{\mathbf{C}}^\alpha \oplus \mathfrak{g}_{\mathbf{C}}^{-\alpha})$ . The projection  $\mathfrak{m}_{\mathbf{C}} \rightarrow \mathfrak{m}$  which sends a vector to its real part induces  $T$ -equivariant  $\mathbf{R}$ -linear isomorphisms  $\mathfrak{g}_{\mathbf{C}}^\alpha \rightarrow \mathfrak{m}^\alpha$  for all  $\alpha \in \mathcal{R}_M^+$ . We endow  $\mathfrak{m}$  with the  $T$ -invariant complex structure provided by this isomorphism (which depends on the choice of the positive weights  $\mathcal{R}_M^+$ ).

**1.5.1. Theorem.** *Let  $D$  be a  $(\sigma, \tau)$ -twisted operator on  $M$  and let  $i_D: R(H, \sigma - \tau) \rightarrow R(G, \sigma)$  be the induction map determined by  $D$ . Then*

$$j_G^* i_D(a) = \sum_{w \in W^H} w \left( \frac{j_H^*(\mathbf{e}(D)a)}{\prod_{\alpha \in \mathcal{R}_M} (1 - e^\alpha)} \right)$$

for all  $a \in R(H, \sigma - \tau)$ .

*Proof.* First assume that  $\sigma = 0$ . Then  $D$  is a  $G \times H^{(\tau)}$ -equivariant operator

$$D: \Gamma(G, G \times U^0) \rightarrow \Gamma(G, G \times U^1)$$

that is transversely elliptic with respect to  $H^{(\tau)}$ , where  $U^0$  and  $U^1$  are  $H^{(\tau)}$ -modules of level 1. Choose a representative  $\dot{w} \in N_G(T)$  for each  $w \in W^H$ . The map  $W^H \rightarrow M^T$  defined by  $w \mapsto \dot{w}H/H$  is a bijection. Let  $V$  be an  $H^{(\tau)}$ -module of level  $-1$  and let  $\chi_V: H^{(\tau)} \rightarrow \mathbf{C}$  be its character. Let  $\chi$  be the character of the virtual  $G$ -module  $i_D([V]) = \text{index}(D_{V^*})$ . Let  $t$  be a generic element of  $T$ . By [3, §II.5],  $\chi(t) = \sum_{w \in W^H} \chi_w(t)$ , where

$$\chi_w(t) = \frac{\text{trace}(\kappa^0(w^{-1}(t))) - \text{trace}(\kappa^1(w^{-1}(t)))}{|\det_{\mathbf{R}}(1 - \eta(w^{-1}(t^{-1})))|},$$

and  $\kappa^p$  is the representation  $H \rightarrow \mathbf{GL}(U^p \otimes V)$  for  $p = 0$  or  $1$ . We have

$$\text{trace}(\kappa^p(w^{-1}(t))) = \chi_{U^p}(w^{-1}(t)) \chi_V(w^{-1}(t)),$$

and arguing as in [3, §II.5] we find

$$\det_{\mathbf{R}}(1 - \eta(w^{-1}(t^{-1}))) = \prod_{\alpha \in \mathcal{R}_M^+} (1 - t^{w(\alpha)})(1 - t^{-w(\alpha)}) > 0.$$

Therefore

$$\chi(t) = \sum_{w \in W^H} \frac{[\chi_{U^0}(w^{-1}(t)) - \chi_{U^1}(w^{-1}(t))] \chi_V(e^{w^{-1}(\xi)})}{\prod_{\alpha \in \mathcal{R}_M} (1 - t^{w(\alpha)})}.$$

This identity, valid for generic  $t$ , amounts to an identity in the fraction field of the representation ring  $R(T)$ , namely

$$j_G^* i_D([V]) = \sum_{w \in W^H} \frac{w j_H^*([(U^0] - [U^1]) \cdot [V])}{\prod_{\alpha \in \mathcal{R}_M} (1 - e^{w(\alpha)})},$$

from which the assertion follows immediately. The case of a general extension  $\sigma$  is handled by replacing  $G$  with  $G^{(\sigma)}$ ,  $H$  with  $H^{(\sigma)}$ , and  $H^{(\tau)}$  with  $H^{(\sigma)} \times_H H^{(\tau)}$ . QED

The following example is well-known; see e.g. [30, § 2].

**1.5.2. Example.** Let  $\sigma = \tau = 0$ . The exterior algebra  $\Lambda(\mathfrak{m}_{\mathbb{C}})$  is a  $\mathbb{Z}/2\mathbb{Z}$ -graded  $H$ -equivariant  $\mathbf{Cl}(\mathfrak{m}_{\mathbb{C}})$ -module. The corresponding Dirac operator defined with respect to the Levi-Civita connection (see § 1.4) is the Hodge-de Rham operator

$$D = d + d^*: \Gamma(M, \Lambda^{\text{even}}(T_{\mathbb{C}}^* M)) \rightarrow \Gamma(M, \Lambda^{\text{odd}}(T_{\mathbb{C}}^* M)),$$

where  $d^*$  is the adjoint of the exterior derivative  $d$  with respect to the Riemannian metric on  $M$ . Also,  $\mathbf{e}(D) = \zeta^*([\sigma(D)]) = \lambda_{-1}([\mathfrak{m}_{\mathbb{C}}])$ , and hence

$$j_H^*(\mathbf{e}(D)) = \lambda_{-1}(j_H^*([\mathfrak{m}_{\mathbb{C}}])) = \prod_{\alpha \in \mathcal{R}_M^+} (1 - e^{\alpha})(1 - e^{-\alpha}).$$

This expression cancels against the denominator in Theorem 1.5.1 and therefore

$$j_G^* i_D(a) = \sum_{w \in W^H} w(j_H^*(a))$$

for all  $a \in R(H)$ . In particular

$$i_!(\mathbf{e}(D)) = i_D(1) = \sum_{w \in W^H} 1 = |W_G/W_H|$$

is the Euler characteristic of  $M$ .

## 2. TWISTED $\mathbf{Spin}^c$ -INDUCTION: THE CHARACTER FORMULA

Among all twisted equivariant elliptic differential operators on a maximal-rank homogeneous space  $M = G/H$  there is a preferred one, the *twisted  $\mathbf{Spin}^c$  Dirac operator*  $\mathcal{D}_M = \mathcal{D}$ . (Indeed, every oriented Riemannian manifold has a uniquely defined natural twisted  $\mathbf{Spin}^c$  Dirac operator.) Thanks to the Thom isomorphism every elliptic induction map can be expressed in terms of  $\mathcal{D}$  (Theorem 2.1.2 below). In this section we develop the properties of twisted  $\mathbf{Spin}^c$ -induction up to the “Weyl character formula” (Theorem 2.2.4).

*Notation.* We retain the hypotheses and the notational conventions stated at the beginning of § 1 and § 1.5. In addition we define

$$\rho_M = \rho_G - \rho_H = \frac{1}{2} \sum_{\alpha \in \mathcal{R}_M^+} \alpha \in \frac{1}{2} \mathcal{X}(T).$$

### 2.1. The Thom isomorphism. Let

$$\mathbf{Spin}^c(\mathfrak{m}) = (\mathbf{U}(1) \times \mathbf{Spin}(\mathfrak{m}))/K$$

be the complex spinor group of  $\mathfrak{m} = T_1 M$ . Here  $K \cong \mathbf{Z}/2\mathbf{Z}$  denotes the central subgroup  $\{(1, 1), (-1, x)\}$  of  $\mathbf{U}(1) \times \mathbf{Spin}(\mathfrak{m})$ , with  $x$  being the nontrivial element in the kernel of the double cover  $\mathbf{Spin}(\mathfrak{m}) \rightarrow \mathbf{SO}(\mathfrak{m})$ . The central extension

$$1 \longrightarrow \mathbf{U}(1) \longrightarrow \mathbf{Spin}^c(\mathfrak{m}) \longrightarrow \mathbf{SO}(\mathfrak{m}) \longrightarrow 1 \quad (2.1)$$

pulls back via the tangent representation  $\eta: H \rightarrow \mathbf{SO}(\mathfrak{m})$  to a central extension

$$\omega = \omega_M: 1 \longrightarrow \mathbf{U}(1) \longrightarrow H \times_{\mathbf{SO}(\mathfrak{m})} \mathbf{Spin}^c(\mathfrak{m}) \longrightarrow H \longrightarrow 1, \quad (2.2)$$

which we call the *orientation system* of  $M$ . (The orientation system is trivial if and only if  $\eta$  lifts to a representation  $H \rightarrow \mathbf{Spin}^c(\mathfrak{m})$ , which is equivalent to  $M$  having a  $G$ -invariant  $\mathbf{Spin}^c$ -structure, i.e. an orientation in equivariant K-theory. See Appendix C.)

Let us write elements of  $\mathbf{Spin}^c(\mathfrak{m})$  as equivalence classes  $[z, g]$  with  $z \in \mathbf{U}(1)$  and  $g \in \mathbf{Spin}(\mathfrak{m})$ . The involution  $f([z, g]) = [z^{-1}, g]$  of  $\mathbf{Spin}^c(\mathfrak{m})$  restricts to the deck transformation of the covering map  $\mathbf{Spin}(\mathfrak{m}) \rightarrow \mathbf{SO}(\mathfrak{m})$  and therefore induces the identity on  $\mathbf{SO}(\mathfrak{m})$ . The involution  $\eta^*(f)$  of  $H^{(\omega)}$  therefore defines an equivalence between the orientation system  $\omega_M$  and its opposite, and hence an isomorphism of  $R(H)$ -modules

$$R(H, \omega_M) \cong R(H, -\omega_M),$$

which we will use to identify these two modules. (See Remark 1.2.1.)

**2.1.1. Remark.** Let  $\text{Ext}(G, \mathbf{U}(1))$  be the group of equivalence classes of central extensions of  $G$  by  $\mathbf{U}(1)$ . There is an isomorphism of abelian groups

$$\text{Ext}(G, \mathbf{U}(1)) \cong H_G^3(\text{pt}, \mathbf{Z}).$$

(See [4, §6] or [38, §2].) Under the isomorphism  $H_G^3(M, \mathbf{Z}) \cong H_H^3(\text{pt}, \mathbf{Z})$ , the class of  $\omega_M$  in  $\text{Ext}(H, \mathbf{U}(1)) \cong H_H^3(\text{pt}, \mathbf{Z})$  corresponds to the integral equivariant Stiefel-Whitney class  $W_G^3(M) \in H_G^3(M, \mathbf{Z})$ , which is a 2-torsion element. If  $G$  is simply connected, the forgetful map  $H_G^3(M, \mathbf{Z}) \rightarrow H^3(M, \mathbf{Z})$  is injective (see [21, Lemma 3.3]), so  $M$  is  $\mathbf{Spin}^c$  if and only if it is  $G$ -invariantly  $\mathbf{Spin}^c$ .

Let  $S = S^0 \oplus S^1$  be the spinor module of the Clifford algebra  $\mathbf{Cl}(\mathfrak{m}_{\mathbf{C}})$ . This is a  $\mathbf{Z}/2\mathbf{Z}$ -graded  $H^{(\omega)}$ -equivariant Clifford module, and the  $H^{(\omega)}$ -action is of level 1. The associated Dirac operator

$$\mathcal{D} = \mathcal{D}_M: \Gamma(G^{(\sigma)}, G^{(\sigma)} \times S^0) \longrightarrow \Gamma(G^{(\sigma)}, G^{(\sigma)} \times S^1)$$

is a  $(\sigma, \omega)$ -twisted equivariant elliptic operator (see §1.4), which we refer to as the *twisted  $\mathbf{Spin}^c$  Dirac operator* on  $M$ . We denote the corresponding induction map by

$$i_* = i_{\mathcal{D}}: R(H, \sigma + \omega_M) \longrightarrow R(G, \sigma).$$

Let  $\pi: T^*M \rightarrow M$  be the cotangent bundle projection. The Thom isomorphism theorem (see e.g. [9]) states that for every central  $\mathbf{U}(1)$ -extension  $\tau$  of  $H$  the map

$$\zeta_*: R(H, \tau) \cong K_G^*(M, \tau) \longrightarrow K_G^*(T^*M, T^\times M, \tau + \omega_M)$$

defined by  $\zeta_*(a) = \mathbf{th}(\mathcal{D})\pi^*(a)$  is an isomorphism of  $R(H)$ -modules. Here  $\mathbf{th}(\mathcal{D})$  is the equivariant Thom class (1.8) of  $\mathcal{D}$ . Note that the equivariant Euler class of  $\mathcal{D}$  is

$$\mathbf{e}(\mathcal{D}) = \zeta^*(\mathbf{th}(\mathcal{D})) = \zeta^*\zeta_*(1) \in R(H, \omega_M).$$

It was observed by Atiyah and Singer that the Thom isomorphism theorem reduces the index theorem for general elliptic operators to the case of the Dirac operator. In exactly the same way the Thom isomorphism enables us to express every induction map defined by a twisted elliptic operator in terms of twisted  $\mathbf{Spin}^c$ -induction.

**2.1.2. Theorem.** *Let  $D$  be a  $(\sigma, \tau)$ -twisted operator on  $M$ . There is a unique element  $a_D \in R(H, \tau + \omega_M)$  such that  $\mathbf{e}(D) = \mathbf{e}(\mathcal{D})a_D$ , namely  $a_D = (\zeta_*)^{-1}(\mathbf{th}(D))$ . This element has the property that  $i_D(a) = i_*(a_D a)$  for all  $a \in R(H, \sigma - \tau)$ .*

*Proof.* Put  $a_D = (\zeta_*)^{-1}(\mathbf{th}(D))$ . Then, by definition of  $\zeta_*$ ,  $\mathbf{th}(D) = \mathbf{th}(\mathcal{D})\pi^*(a_D)$ . Pulling back to the zero section gives

$$\mathbf{e}(D) = \zeta^*(\mathbf{th}(D)) = \zeta^*(\mathbf{th}(\mathcal{D}))a_D = \mathbf{e}(\mathcal{D})a_D.$$

It follows from Lemma 1.2.2(iii) that  $a_D$  is the only class that satisfies this identity. Using this and applying Theorem 1.3.4(i) twice we find that

$$i_D(a) = i_!(\mathbf{e}(D)a) = i_!(\mathbf{e}(\mathcal{D})a_D a) = i_D(a_D a)$$

for all  $a \in R(H, \sigma - \tau)$ . QED

**2.1.3. Example.** Let  $V$  be an  $H^{(\tau - \omega_M)}$ -module of level 1. The twisted Dirac operator  $D$  defined by the  $H^{(\tau)}$ -equivariant Clifford module  $S \otimes V$  is a  $(\sigma, \tau)$ -twisted operator. (See § 1.4.) The Thom class of  $D$  is

$$\mathbf{th}(D) = [\text{symbol}(D)] = [\text{symbol}(\mathcal{D})]\pi^*([V]) = \mathbf{th}(\mathcal{D})\pi^*([V]),$$

so the class  $a_D$  is equal to

$$(\zeta_*)^{-1}(\mathbf{th}(D)) = \pi_*(\mathbf{th}(D)) = \pi_*(\mathbf{th}(\mathcal{D})\pi^*([V])) = \pi_*(\mathbf{th}(\mathcal{D}))[V] = [V].$$

Hence  $i_D(a) = i_*([V]a)$  for all  $a \in R(H, \sigma - \tau)$  by Theorem 2.1.2.

**2.2. The character formula.** The centre of  $\mathbf{Spin}^c(\mathfrak{m})$  is  $\mathbf{U}(1)$  and its commutator subgroup is  $\mathbf{Spin}(\mathfrak{m})$ , the universal covering group of  $\mathbf{SO}(\mathfrak{m})$ . We therefore have a natural infinitesimal splitting  $\mathfrak{spin}^c(\mathfrak{m}) = \mathfrak{u}(1) \oplus \mathfrak{so}(\mathfrak{m})$  of the extension (2.1). Pulling back to  $H$  we obtain an infinitesimal splitting of the orientation system  $\omega$ , and then restricting to the maximal torus  $T$  of  $H$  we get an infinitesimal splitting of the extension

$$1 \longrightarrow \mathbf{U}(1) \longrightarrow T^{(\omega)} \longrightarrow T \longrightarrow 1,$$

where  $T^{(\omega)} \subseteq H^{(\omega)}$  is the inverse image of  $T$ . Dually we have a rational splitting  $q: \mathcal{X}(U(1))_{\mathbf{Q}} \rightarrow \mathcal{X}(T^{(\omega)})_{\mathbf{Q}}$  of the exact sequence of character groups

$$0 \longrightarrow \mathcal{X}(T) \longrightarrow \mathcal{X}(T^{(\omega)}) \longrightarrow \mathcal{X}(\mathbf{U}(1)) \longrightarrow 0.$$

We identify  $\mathcal{X}(T)$  with its image in  $\mathcal{X}(T^{(\omega)})$  and  $\mathcal{X}(\mathbf{U}(1))$  with its image  $q(\mathcal{X}(\mathbf{U}(1)))$  in  $\mathcal{X}(T^{(\omega)})_{\mathbf{Q}}$ . Let  $\varepsilon_0 = \text{id}_{U(1)}$  be the standard generator of  $\mathcal{X}(\mathbf{U}(1))$ . Since  $\mathbf{Spin}(\mathfrak{m})$  is a double covering of  $\mathbf{SO}(\mathfrak{m})$ , we have  $\varepsilon_0 \in \frac{1}{2}\mathcal{X}(T^{(\omega)})$ .

In the next lemma we compute the Euler class  $\mathbf{e}(\mathcal{D}) \in R(H, \omega_M)$ , or rather its restriction to the maximal torus  $T^{(\omega)}$  of  $H^{(\omega)}$ . For ease of notation we denote the inclusion  $T^{(\omega)} \rightarrow H^{(\omega)}$  by  $j_H$ .

- 2.2.1. Lemma.** (i)  $\varepsilon_0 - \rho_M \in \mathcal{X}(T^{(\omega)})$ .  
(ii)  $\mathcal{X}(T^{(\omega)}) = \mathcal{X}(T) \oplus \mathbf{Z} \cdot (\varepsilon_0 - \rho_M)$ .  
(iii)  $R(T, \omega) = R(T) \cdot e^{\varepsilon_0 - \rho_M}$ .  
(iv)  $j_H^*(\mathbf{e}(\mathcal{D})) = e^{\varepsilon_0 - \rho_M} \prod_{\alpha \in \mathcal{R}_M^+} (1 - e^\alpha) = e^{\varepsilon_0} \prod_{\alpha \in \mathcal{R}_M^+} (e^{-\alpha/2} - e^{\alpha/2})$ .

*Proof.* Let  $\mathcal{R}_M^+ = \{\alpha_1, \alpha_2, \dots, \alpha_l\}$  and put  $\mathfrak{m}_k = \mathfrak{m}^{\alpha_k}$ . The group

$$\mathbf{T} = \prod_{k=1}^l \mathbf{SO}(\mathfrak{m}_k)$$

is a maximal torus of  $\mathbf{SO}(\mathfrak{m})$  that contains the torus  $\eta(T)$ . Let  $\varepsilon_k$  be the weight of the  $\mathbf{T}$ -action on  $\mathfrak{m}_k$ . (Recall our convention whereby we identify  $\mathfrak{m}_k$  with the root space  $\mathfrak{g}^{\alpha_k}$ , so turning  $\mathfrak{m}$  into a unitary  $\mathbf{T}$ -module.) Let  $\hat{\mathbf{T}}$  be the inverse image of  $\mathbf{T}$  in  $\mathbf{Spin}^c(\mathfrak{m})$ . Using [8, Ch. VI, Planche IV] we compute the character groups of  $\mathbf{T}$  and  $\hat{\mathbf{T}}$  to be

$$\mathcal{X}(\mathbf{T}) = \bigoplus_{k=1}^l \mathbf{Z}\varepsilon_k, \quad \mathcal{X}(\hat{\mathbf{T}}) = \mathbf{Z}\hat{\varepsilon} \oplus \bigoplus_{k=1}^l \mathbf{Z}\varepsilon_k,$$

where  $\hat{\varepsilon} = \varepsilon_0 - \frac{1}{2} \sum_{k=1}^l \varepsilon_k$ . Because  $T$  acts with weight  $\alpha_k$  on  $\mathfrak{m}_k$ , the homomorphisms

$$\eta^*: \mathcal{X}(\mathbf{T}) \rightarrow \mathcal{X}(T), \quad \hat{\eta}^*: \mathcal{X}(\hat{\mathbf{T}}) \rightarrow \mathcal{X}(T^{(\omega)})$$

induced by  $\eta: T \rightarrow \mathbf{T}$  and its lift  $\hat{\eta}: T^{(\omega)} \rightarrow \hat{\mathbf{T}}$  are given by

$$\eta^*(\varepsilon_k) = \alpha_k, \quad \hat{\eta}^*(\hat{\varepsilon}) = \varepsilon_0 - \rho_M.$$

By definition  $T^{(\omega)}$  is the pullback  $T \times_{\mathbf{T}} \hat{\mathbf{T}}$ , so dually its character group  $\mathcal{X}(T^{(\omega)})$  is the pushout

$$\mathcal{X}(T^{(\omega)}) \cong (\mathcal{X}(T) \oplus \mathcal{X}(\hat{\mathbf{T}})) / \mathcal{X}(\mathbf{T}).$$

Hence  $\mathcal{X}(T^{(\omega)})$  is generated by  $\mathcal{X}(T)$  and  $\varepsilon_0 - \rho_M$ , which proves (i) and (ii). Let  $\lambda \in \mathcal{X}(T^{(\omega)})$  and write  $\lambda = \mu + k(\varepsilon_0 - \rho_M)$  with  $\mu \in \mathcal{X}(T)$  and  $k \in \mathbf{Z}$ . Then the element  $e^\lambda \in \mathbf{Z}[\mathcal{X}(T^{(\omega)})] \cong R(T^{(\omega)})$  is of level 1 if and only if  $k = 1$ . Thus the  $R(T)$ -module  $R(T, \omega)$  is spanned by the element  $e^{\varepsilon_0 - \rho_M}$ , which proves (iii). According to (1.9), the Euler class of  $\mathcal{D}$  is

$$\mathbf{e}(\mathcal{D}) = \hat{\eta}^*([S^0] - [S^1]) \in R(H, \omega). \quad (2.3)$$

To calculate its restriction to  $R(T, \omega)$ , we must describe the  $T$ -action on the half-spin representations  $S^0$  and  $S^1$ . As vector spaces,

$$S = \Lambda_{\mathbf{C}}(\mathfrak{m}), \quad S^0 = \Lambda_{\mathbf{C}}^{\text{even}}(\mathfrak{m}), \quad S^1 = \Lambda_{\mathbf{C}}^{\text{odd}}(\mathfrak{m}).$$

For  $1 \leq k \leq l$ , let  $v_k \in \mathfrak{m}_k$  be a vector of unit length. Then the products

$$v_{\mathbf{i}} = v_{i_1} \wedge \dots \wedge v_{i_k} \quad (2.4)$$

where  $\mathbf{i} = (i_1, i_2, \dots, i_k)$  ranges over all increasing multi-indices  $\mathbf{i}$  of length  $|\mathbf{i}| = k \in \{0, 1, \dots, l\}$ , form an orthonormal basis of  $\Lambda_{\mathbf{C}}(\mathfrak{m})$ . Let  $\tilde{\mathbf{T}}$  be the inverse image in  $\mathbf{Spin}(\mathfrak{m})$  of the maximal torus  $\mathbf{T}$ . Its character group is

$$\mathcal{X}(\tilde{\mathbf{T}}) = \mathbf{Z}\tilde{\varepsilon} + \bigoplus_{k=1}^l \mathbf{Z}\varepsilon_k,$$

where  $\tilde{\varepsilon} = \frac{1}{2} \sum_{k=1}^l \varepsilon_k$ . The vector  $v_i$  is a weight vector of the  $\hat{\mathbf{T}}$ -action with weight  $-\tilde{\varepsilon} + \varepsilon_i$ , where  $\varepsilon_i = \varepsilon_{i_1} + \cdots + \varepsilon_{i_k}$ . (Cf. [8, § VIII.13.4].) The centre  $\mathbf{U}(1)$  of  $\mathbf{Spin}^c(\mathfrak{m})$  acts by scalar multiplication, so the weight of  $v_i$  for the  $\hat{\mathbf{T}}$ -action is  $\varepsilon_0 - \tilde{\varepsilon} + \varepsilon_i = \hat{\varepsilon} + \varepsilon_i$ . Hence the weight of  $v_i$  for the  $T$ -action is

$$\hat{\eta}(\hat{\varepsilon} + \varepsilon_i) = \varepsilon_0 - \rho_M + \alpha_i. \quad (2.5)$$

Thus the class of  $\Lambda_{\mathbb{C}}^k(\mathfrak{m})$  in  $R(T, \omega)$  is  $e^{\varepsilon_0 - \rho_M} \sum_{|\mathbf{i}|=k} e^{\alpha_i}$ , and hence

$$j_H^*(\mathbf{e}(\mathcal{D})) = e^{\varepsilon_0 - \rho_M} \sum_{k=1}^l (-1)^k \sum_{|\mathbf{i}|=k} e^{\alpha_i} = e^{\varepsilon_0 - \rho_M} \prod_{k=1}^l (1 - e^{\alpha_k}),$$

which establishes (iv). QED

**2.2.2. Remark.** Let  $\mathbf{Z}[\rho_M + \mathcal{X}(T)]$  be the free abelian group generated by the affine lattice  $\rho_M + \mathcal{X}(T)$  in  $\mathcal{X}(T)_{\mathbb{Q}}$ . The basis elements of  $\mathbf{Z}[\rho_M + \mathcal{X}(T)]$  are of the form  $e^{\mu + \rho_M}$  with  $\mu \in \mathcal{X}(T)$ . The group

$$\mathbf{Z}[\rho_M + \mathcal{X}(T)] = \mathbf{Z}[\mathcal{X}(T)] e^{\rho_M} \cong R(T) \cdot e^{\rho_M}$$

is not a ring, but a module over the ring  $\mathbf{Z}[\mathcal{X}(T)] \cong R(T)$ . By Lemma 2.2.1(iii), the affine map  $\rho_M + \mathcal{X}(T) \rightarrow \mathcal{X}(T^{(\omega)})$  defined by  $\lambda \mapsto \varepsilon_0 + \lambda$  induces an isomorphism

$$R(T) \cdot e^{\rho_M} \xrightarrow{\cong} R(T, \omega), \quad a \mapsto e^{\varepsilon_0} \cdot a$$

of  $R(T)$ -modules, which we will call the *level shift*. The preimage of  $j_H^*(\mathbf{e}(\mathcal{D}))$  under the level shift is the class

$$e^{-\rho_M} \prod_{\alpha \in \mathcal{R}_M^+} (1 - e^{\alpha}) \in \mathbf{Z}[\rho_M + \mathcal{X}(T)].$$

Since  $\rho_M - w(\rho_M)$  is in the root lattice of  $G$  for all  $w \in W_H$ , the  $W_H$ -action on  $\mathcal{X}(T)_{\mathbb{Q}}$  preserves  $\rho_M + \mathcal{X}(T)$ . Because  $\varepsilon_0$  is central, the level shift is  $W_H$ -equivariant and hence restricts to an isomorphism of  $R(H)$ -modules

$$\mathbf{Z}[\rho_M + \mathcal{X}(T)]^{W_H} \xrightarrow{\cong} R(H, \omega),$$

which we also refer to as the level shift. (If  $\rho_M \notin \mathcal{X}(T)$ , i.e. if  $M$  does not have a  $G$ -invariant  $\mathbf{Spin}$ -structure, we can view  $R(H) \oplus R(H, \omega)$  as the representation ring of the double cover  $H \times_{\mathbf{SO}(\mathfrak{m})} \mathbf{Spin}(\mathfrak{m})$  of  $H$ .) In the sequel we will make the identifications

$$R(T, \omega) = R(T) \cdot e^{\rho_M}, \quad R(H, \omega) = (R(T) \cdot e^{\rho_M})^{W_H}.$$

Formally this amounts to setting  $\varepsilon_0 = 0$ , which has the desirable effect of cleaning up several formulas.

The *antisymmetrizer* of  $W_G$  is the element  $J_G$  of the group ring  $\mathbf{Z}[W_G]$  defined by

$$J_G = \sum_{w \in W_G} \det(w) w. \quad (2.6)$$

Let  $A$  be a  $W_G$ -module and let  $A^{-W_G}$  denote the set of *anti-invariant* elements of  $A$ , i.e. those  $a \in A$  satisfying  $w(a) = \det(w)a$  for all  $w \in W_G$ . Then  $J_G$  defines a

$\mathbf{Z}[W_G]$ -linear operator  $J_G: A \rightarrow A^{-W_G}$ , and we have  $J_G(a) = |W_G|a$  for all anti-invariant  $a \in A$ . Similarly, we have the antisymmetrizer  $J_H \in \mathbf{Z}[W_H]$  with respect to the Weyl group  $W_H$ . We also define a *relative* antisymmetrizer  $J_M \in \mathbf{Z}[W_G]$  by

$$J_M = \sum_{w \in W^H} \det(w)w.$$

The fact that each  $w \in W$  can be written uniquely as a product  $w'w''$  with  $w' \in W^H$  and  $w'' \in W_H$  implies that

$$J_G = J_M J_H. \quad (2.7)$$

Recall that the  $W_G$ -anti-invariant element  $J_G(e^{\rho_G})$  of  $\mathbf{Z}[\frac{1}{2}\mathcal{X}(T)]$  is equal to the *Weyl denominator*

$$\mathbf{d}_G = e^{\rho_G} \prod_{\alpha \in \mathcal{R}_G^+} (1 - e^{-\alpha}). \quad (2.8)$$

(See e.g. [8, § VI.3, Proposition 2].) The *duality homomorphism*  $R(H, \tau) \rightarrow R(H, -\tau)$  is the map  $b \mapsto b^*$  defined on generators by  $[V]^* = [V^*]$ .

**2.2.3. Lemma.** *Let  $\mathcal{D}_{G/T}$  and  $\mathcal{D}_{H/T}$  be the twisted  $\mathbf{Spin}^c$  Dirac operators of the flag varieties  $G/T$ , resp.  $H/T$ . We have the identities*

$$\mathbf{d}_G = \mathbf{e}(\mathcal{D}_{G/T})^*, \quad j_H^*(\mathbf{e}(\mathcal{D}_M)^*) = \mathbf{d}_G / \mathbf{d}_H, \quad \mathbf{e}(\mathcal{D}_{G/T}) = j_H^*(\mathbf{e}(\mathcal{D}_{G/H}))\mathbf{e}(\mathcal{D}_{H/T}).$$

*Proof.* This follows immediately from Lemma 2.2.1(iv) and (2.8) (after applying a level shift; see Remark 2.2.2). QED

A version of the following character formula was stated by Bott [7, p. 179], but his hypotheses and proof are not entirely correct. Incorporating the  $\omega$ -twist resolves the problems in Bott's treatment. A version for Lie algebras (which does not require any twists) can be found in [13]. Note that the character formula depends on the choice of the positive roots. The induction map  $i_*$ , however, is independent of this choice.

**2.2.4. Theorem.** *Let  $\mathcal{D}_M$  be the twisted  $\mathbf{Spin}^c$  Dirac operator on  $M$  and let  $i_*: R(H, \sigma + \omega_M) \rightarrow R(G, \sigma)$  be the associated induction map. Then*

$$j_G^* i_*(a) = \frac{J_M(\mathbf{d}_H j_H^*(a))}{\mathbf{d}_G}$$

for all  $a \in R(H, \sigma + \omega_M)$ .

*Proof.* It follows from Lemma 2.2.1(iv) that

$$j_H^*(\mathbf{e}(\mathcal{D})\mathbf{e}(\mathcal{D})^*) = \prod_{\alpha \in \mathcal{R}_M} (1 - e^\alpha).$$

Substituting this expression into the formula of Theorem 1.5.1 yields

$$j_G^* i_*(a) = J_M \left( j_H^* \left( \frac{a}{\mathbf{e}(\mathcal{D})^*} \right) \right)$$

for all  $a \in R(H, \sigma + \omega_M)$ . Using Lemma 2.2.3 gives

$$w j_H^* \left( \frac{a}{\mathbf{e}(\mathcal{D})^*} \right) = w \left( \frac{\mathbf{d}_H j_H^*(a)}{\mathbf{d}_G} \right) = \frac{\det(w) w(\mathbf{d}_H j_H^*(a))}{\mathbf{d}_G}$$

for all  $w \in W^H$ , which proves the result. QED



Let us explain the extent to which this formula is parallel to Weyl's character formula. For simplicity let  $\sigma = 0$ . The following result generalizes a well-known result for semisimple simply connected groups. The proof is in Appendix B.

- 2.2.5. Proposition.** (i) *The  $W_G$ -action on  $R(T^{(\omega_{G/T})})$  preserves  $R(T, \omega_{G/T})$ .*  
(ii) *The set of anti-invariants  $R(T, \omega_{G/T})^{-W_G}$  is a free  $R(G)$ -module of rank 1 generated by  $\mathbf{d}_G$ .*  
(iii) *The elements  $J_G(e^\lambda)$ , with  $\lambda \in \rho_G + \mathcal{X}(T)$  strictly dominant, form a basis of the  $\mathbf{Z}$ -module  $R(T, \omega_{G/T})^{-W_G}$ .*

Applying this to  $G$  and  $H$  and using the equivalence  $\omega_{G/H} + \omega_{H/T} \sim \omega_{G/T}$  (see Lemma 3.1.1 below), we can interpret the character formula for twisted  $\mathbf{Spin}^c$ -induction as a three-step process,

$$R(H, \omega_{G/H}) \xrightarrow[\cong]{\mathbf{d}_H} R(T, \omega_{G/T})^{-W_H} \xrightarrow{J_M} R(T, \omega_{G/T})^{-W_G} \xrightarrow[\cong]{\mathbf{d}_G} R(G).$$

### 3. FURTHER PROPERTIES OF TWISTED $\mathbf{Spin}^c$ -INDUCTION

We continue our discussion of the twisted induction map defined by means of the twisted  $\mathbf{Spin}^c$  Dirac operator on the homogeneous space  $M = G/H$ . We retain the notational conventions of §§1 and 2. We show that twisted  $\mathbf{Spin}^c$ -induction is functorial with respect to the subgroup  $H$  (Theorem 3.1.2). If  $M$  has an invariant  $\mathbf{Spin}^c$ -structure (which is not always the case), then there is a transparent relationship between  $\mathbf{Spin}^c$ -induction and twisted  $\mathbf{Spin}^c$ -induction (Theorem 3.2.1). Twisted induction of irreducible modules behaves according to a “Borel-Weil-Bott formula” (Theorem 3.3.1). Twisted  $\mathbf{Spin}^c$ -induction gives rise to a bilinear pairing between twisted representation modules. These modules are twisted equivariant K-groups, and the pairing is analogous to an intersection pairing in ordinary equivariant cohomology. Under a mild condition on  $G$  the pairing is nonsingular (Theorem 3.4.2), which enables us to characterize all twisted induction maps from  $H$  to  $G$  (Theorem 3.4.3).

**3.1. Twisted induction in stages.** Twisted  $\mathbf{Spin}^c$ -induction is functorial with respect to the subgroup  $H$ . Consider two closed connected subgroups  $H_2 \subseteq H_1$  of  $G$ , both of which contain  $T$ . We have a diagram of inclusions

$$\begin{array}{ccc} & & H_1 \\ & \nearrow k & \downarrow i_1 \\ H_2 & \xrightarrow{i_2} & G. \end{array}$$

Let  $\mathfrak{m}_1 = \mathfrak{g}/\mathfrak{h}_1$ ,  $\mathfrak{m}_2 = \mathfrak{g}/\mathfrak{h}_2$  and  $\mathfrak{n} = \mathfrak{h}_1/\mathfrak{h}_2$ . Being a sum of root spaces,  $\mathfrak{m}_2$  is naturally isomorphic to the orthogonal direct sum  $\mathfrak{m}_1 \oplus \mathfrak{n}$ . We have three orientation systems,

$$\begin{aligned} \omega_{G/H_p} : 1 &\longrightarrow \mathbf{U}(1) \longrightarrow H_p \times_{\mathbf{SO}(\mathfrak{m}_p)} \mathbf{Spin}^c(\mathfrak{m}_p) \longrightarrow H_p \longrightarrow 1 \quad (p = 1, 2), \\ \omega_{H_1/H_2} : 1 &\longrightarrow \mathbf{U}(1) \longrightarrow H_2 \times_{\mathbf{SO}(\mathfrak{n})} \mathbf{Spin}^c(\mathfrak{n}) \longrightarrow H_2 \longrightarrow 1. \end{aligned}$$

**3.1.1. Lemma.** *The natural homomorphism  $f : \mathbf{SO}(\mathfrak{m}_1) \times \mathbf{SO}(\mathfrak{n}) \rightarrow \mathbf{SO}(\mathfrak{m}_2)$  induces an equivalence of extensions  $f^* : k^*(\omega_{G/H_1}) + \omega_{H_1/H_2} \xrightarrow{\sim} \omega_{G/H_2}$ .*

*Proof.* For  $p = 0$  or  $1$ , put  $\mathbf{G}_p = \mathbf{SO}(\mathfrak{m}_p)$  and  $\hat{\mathbf{G}}_p = \mathbf{Spin}^c(\mathfrak{m}_p)$ . Put  $\mathbf{G} = \mathbf{SO}(\mathfrak{n})$  and  $\hat{\mathbf{G}} = \mathbf{Spin}^c(\mathfrak{n})$ . The homomorphism  $f: \mathbf{G}_1 \times \mathbf{G} \rightarrow \mathbf{G}_2$  lifts to a morphism of  $\mathbf{U}(1)$ -extensions of  $\mathbf{G}_1$ ,

$$(\hat{\mathbf{G}}_1 \times \hat{\mathbf{G}})/K \longrightarrow \hat{\mathbf{G}}_2, \quad (3.1)$$

where  $K$  is a copy of  $\mathbf{U}(1)$  anti-diagonally embedded in  $\hat{\mathbf{G}}_2 \times \hat{\mathbf{G}}$ . The morphism (3.1) is an isomorphism by the five-lemma. Pulling the extension  $H_1^{(\omega_{G/H_1})}$  of  $H_1$  back to  $H_2$  gives the  $\mathbf{U}(1)$ -extension

$$H_2 \times_{H_1} (H_1 \times_{\mathbf{G}_1} \hat{\mathbf{G}}_1) \cong H_2 \times_{\mathbf{G}_1} \hat{\mathbf{G}}_1$$

of  $H_2$ . Adding this extension to  $H_2^{(\omega_{H_1/H_2})}$  gives the extension

$$\begin{aligned} [(H_2 \times_{\mathbf{G}_1} \hat{\mathbf{G}}_1) \times_{H_2} (H_2 \times_{\mathbf{G}} \hat{\mathbf{G}})]/K &\cong [(H_2 \times_{\mathbf{G}_1} \hat{\mathbf{G}}_1) \times_{\mathbf{G}} \hat{\mathbf{G}}]/K \\ &\cong H_2 \times_{\mathbf{G}_1 \times \mathbf{G}} [(\hat{\mathbf{G}}_1 \times \hat{\mathbf{G}})/K] \end{aligned}$$

of  $H_2$ , which is isomorphic to  $H_2 \times_{\mathbf{G}_2} \hat{\mathbf{G}}_2$  by (3.1). QED

For the sake of brevity we will write this equivalence as

$$\omega_{G/H_1} + \omega_{H_1/H_2} \sim \omega_{G/H_2}$$

and use it to make the identification

$$R(H_2, \sigma + \omega_{G/H_2}) = R(H_2, \sigma + \omega_{G/H_1} + \omega_{H_1/H_2}).$$

A version of the next theorem, under the assumption that  $G/H_1$  and  $H_1/H_2$  are  $\mathbf{Spin}$ , was proved in [34, §4].

**3.1.2. Theorem.** (i) *The Euler class of the twisted  $\mathbf{Spin}^c$  Dirac operator is multiplicative in the sense that  $\mathbf{e}(\mathcal{D}_{G/H_2}) = k^*(\mathbf{e}(\mathcal{D}_{G/H_1}))\mathbf{e}(\mathcal{D}_{H_1/H_2})$ .*  
(ii) *Twisted  $\mathbf{Spin}^c$ -induction is functorial in the sense that the diagram*

$$\begin{array}{ccc} R(H_2, \sigma + \omega_{G/H_1} + \omega_{H_1/H_2}) & \xrightarrow{k_*} & R(H_1, \sigma + \omega_{G/H_1}) \\ \parallel & & \downarrow i_{1,*} \\ R(H_2, \sigma + \omega_{G/H_2}) & \xrightarrow{i_{2,*}} & R(G, \sigma) \end{array}$$

*commutes.*

*Proof.* (i) follows from Lemma 2.2.1(iv). Let  $a \in R(H_2, \sigma + \omega_{G/H_2})$ . It follows from (1.2) that the formal pushforward homomorphism is functorial in the sense that  $i_{2,!} = i_{1,!} \circ k_!$ . Hence, using Theorem 1.3.4(i), the  $R(H_1)$ -linearity of the formal pushforward, and the multiplicativity of the Euler class, we obtain

$$\begin{aligned} i_{2,*}(a) &= i_{2,!}(\mathbf{e}(\mathcal{D}_{G/H_2})a) \\ &= i_{1,!}k_!(k^*(\mathbf{e}(\mathcal{D}_{G/H_1}))\mathbf{e}(\mathcal{D}_{H_1/H_2})a) \\ &= i_{1,!}(\mathbf{e}(\mathcal{D}_{G/H_1})k_!(\mathbf{e}(\mathcal{D}_{H_2/H_1})a)) \\ &= i_{1,*}k_*(a) \end{aligned}$$

for all  $a$ , so  $i_{2,*} = i_{1,*} \circ k_*$ . QED

**3.2.  $\mathbf{Spin}^c$  versus twisted  $\mathbf{Spin}^c$ .** We call a character  $\gamma \in \mathcal{X}(H)$  *c-spinorial* if the homomorphism  $\gamma \times \eta: H \rightarrow \mathbf{U}(1) \times \mathbf{SO}(\mathfrak{m})$  lifts to a homomorphism  $\hat{\eta}: H \rightarrow \mathbf{Spin}^c(\mathfrak{m})$ . Such characters  $\gamma$  exist if and only if  $M$  admits a  $G$ -invariant  $\mathbf{Spin}^c$ -structure, and they classify such structures up to equivalence. See Appendix C and [25] for a discussion of invariant  $\mathbf{Spin}^c$ -structures. Let  $\gamma$  be a c-spinorial character. Then we have the untwisted elliptic operator

$$\delta = \delta_\gamma: \Gamma(M, G \times^H S^0) \xrightarrow{\nabla} \Gamma(M, G \times^H (\mathfrak{m} \times S^0)) \xrightarrow{\text{cliff}} \Gamma(M, G \times^H S^1),$$

called the  **$\mathbf{Spin}^c$  Dirac operator** on  $M$  associated with  $\gamma$ . Coupling the Dirac operator with untwisted  $H$ -modules gives rise to an induction map

$$i_\delta = i_{\delta_\gamma}: R(H) \longrightarrow R(G).$$

**3.2.1. Theorem.** *Let  $\gamma$  be a c-spinorial character of  $H$ . Then the  $R(H)$ -module  $R(H, \omega)$  is freely generated by  $e^{\gamma/2}$ , and the diagram*

$$\begin{array}{ccc} R(H) & \xrightarrow{i_\delta} & R(G) \\ e^{\gamma/2} \downarrow \cong & \nearrow i_* & \\ R(H, \omega) & & \end{array}$$

*commutes. Hence  $j_G^* i_\delta(a) = J_M(e^{\gamma/2} \mathbf{d}_H a) / \mathbf{d}_G$  for all  $a \in R(H)$ .*

*Proof.* As in Remark 2.2.2, we identify  $R(H, \omega)$  with the  $R(H)$ -module

$$(R(T) \cdot e^{\rho_M})^{W_H} \cong (\mathbf{Z}[\mathcal{X}(T)] \cdot e^{\rho_M})^{W_H}.$$

By Proposition C.3(ii), the element  $\rho_M - \frac{1}{2}j_H^*(\gamma)$  is in  $\mathcal{X}(T)$ , and therefore  $e^{j_H^*(\gamma)/2}$  generates the  $\mathbf{Z}[\mathcal{X}(T)]$ -module  $\mathbf{Z}[\mathcal{X}(T)] \cdot e^{\rho_M}$ . Being the restriction of a character of  $H$ , the element  $e^{j_H^*(\gamma)/2}$  is invariant under  $W_H$ . Therefore

$$(\mathbf{Z}[\mathcal{X}(T)] \cdot e^{\rho_M})^{W_H} = (\mathbf{Z}[\mathcal{X}(T)] \cdot e^{j_H^*(\gamma)/2})^{W_H} = \mathbf{Z}[\mathcal{X}(T)]^{W_H} \cdot e^{j_H^*(\gamma)/2},$$

which implies that  $e^{\gamma/2}$  generates  $R(H, \omega)$ . Since  $e^{\gamma/2}$  is a unit, this shows that  $R(H, \omega)$  is free. By Proposition C.4(iv), the Euler class of  $\delta_\gamma$  is  $e^{\gamma/2} \mathbf{e}(\mathcal{D})$ . Therefore, by Theorem 2.1.2,  $i_\delta(a) = i_*(e^{\gamma/2} a)$  for all  $a \in R(H)$ . The last assertion now follows from the character formula, Theorem 2.2.4. QED

The best-known special cases of  $\mathbf{Spin}^c$ -induction are  **$\mathbf{Spin}$ -induction** and **holomorphic induction**. The comparison with twisted  $\mathbf{Spin}^c$ -induction is as follows.

**3.2.2. Example.** Suppose that  $M$  possesses a  $G$ -invariant  **$\mathbf{Spin}$ -structure**. This is the case if and only if  $\rho_M \in \mathcal{X}(T)$ . The c-spinorial character of the corresponding  $\mathbf{Spin}^c$ -structure is  $\gamma = 0$ . (See Example C.5.) Therefore, in this case twisted  $\mathbf{Spin}^c$ -induction is the same as  **$\mathbf{Spin}$ -induction**. (However, twisted induction is defined even if  $M$  has no invariant  **$\mathbf{Spin}$ -structure**.) The character formula reads  $j_G^* i_\delta(a) = J_M(\mathbf{d}_H a) / \mathbf{d}_G$  for all  $a \in R(H)$ . In particular, setting  $H = T$  and supposing that  $\rho_G \in \mathcal{X}(T)$ , we find  $j_G^* j_{G, \delta}(a) = J_G(a) / \mathbf{d}_G$  for all  $a \in R(T)$ .

**3.2.3. Example.** Suppose that  $M$  possesses a  $G$ -invariant complex structure. This is the case if and only if  $H$  is the centralizer of a subtorus of  $T$ . The c-spinorial

character for the associated  $\mathbf{Spin}^c$ -structure is given by  $j_H^*(\gamma) = 2\rho_M$ . (See Example C.8.) Therefore holomorphic induction is given by  $i_\delta(a) = i_*(e^{\rho_M}a)$ . Holomorphic induction depends on the choice of the invariant complex structure (in other words, the choice of the basis of the root system), but twisted induction does not. The character formula is  $j_G^*i_\delta(a) = J_M(e^{\rho_M}\mathbf{d}_H a)/\mathbf{d}_G$  for all  $a \in R(H)$ . In particular, setting  $H = T$  gives  $j_G^*j_{G,\delta}(a) = J_G(e^{\rho_G}a)/\mathbf{d}_G$  for all  $a \in R(T)$ , which is the usual Weyl character formula.

**3.3. Irreducibles.** Using functoriality we now compute the pushforward of an irreducible  $H$ -module. A version of the following Borel-Weil-Bott type result for homogeneous  $\mathbf{Spin}$ -manifolds was obtained by Slebarski [34], for Lie algebra representations by Gross et al. [13], and for homogeneous  $\mathbf{Spin}^c$ -manifolds by Landweber [23], [24].

An irreducible  $G^{(\sigma)}$ -module  $V$  is determined up to isomorphism by its highest weight, which is a dominant character  $\lambda \in \mathcal{X}(T^{(\sigma)})$ . If  $V$  is of level 1, then its highest-weight space  $V_\lambda$  is a one-dimensional  $T^{(\sigma)}$ -module of level 1. We will call such a  $\lambda$  a  $G$ -dominant character of level 1, and denote the module  $V$  by  $V_G(\lambda, \sigma)$ . If  $\sigma = 0$ , we write  $V_G(\lambda, \sigma) = V_G(\lambda)$ .

**3.3.1. Theorem.** *Let  $\mu$  be an  $H$ -dominant level 1 character of  $T^{(\sigma+\omega_M)}$ . There exists at most one  $w \in W^H$  such that  $w(\mu + \rho_H) - \rho_G$  is  $G$ -dominant. If no such  $w$  exists, then  $i_*([V_H(\mu, \sigma + \omega_M)]) = 0$ ; if  $w$  exists, then*

$$i_*([V_H(\mu, \sigma + \omega_M)]) = \det(w)[V_G(w(\mu + \rho_H) - \rho_G, \sigma)].$$

*Proof.* The Borel-Weil-Bott theorem for the group  $G$  implies the following assertion: for each  $\kappa \in \mathcal{X}(T)$  there is at most one  $v \in W_G$  such that  $v(\kappa + \rho_G) - \rho_G$  is  $G$ -dominant; if no such  $v$  exists, then  $j_{G,\delta}(e^\kappa) = 0$ ; if  $v$  exists, then

$$j_{G,\delta}(e^\kappa) = \det(v)[V_G(v(\kappa + \rho_G) - \rho_G)].$$

Here  $j_{G,\delta}: R(T) \rightarrow R(G)$  is the holomorphic induction map. (See e.g. [10].) Let us apply this to the group  $G^{(\sigma)}$  and recall that holomorphic induction preserves the level. (This follows from Lemma 1.2.4.) Let us also use the formula  $j_{G,*}(e^\lambda) = j_{G,\delta}(e^{\lambda-\rho_G})$  of Example 3.2.3. The following statement results: for each level 1 character  $\lambda$  of  $T^{(\sigma)}$  there is at most one  $v \in W_G$  such that  $v(\lambda) - \rho_G$  is  $G$ -dominant; if no such  $v$  exists, then  $j_{G,*}(e^\lambda) = 0$ ; if  $v$  exists, then

$$j_{G,*}(e^\lambda) = \det(v)[V_G(v(\lambda) - \rho_G, \sigma)]. \quad (3.2)$$

This statement applies to the group  $H^{(\sigma+\omega_M)}$ . Taking an  $H$ -dominant level 1 character  $\mu$  of  $T^{(\sigma+\omega_M)}$  and putting  $\lambda = \mu + \rho_H$ , we find  $v = 1$  and

$$[V_H(\mu, \sigma + \omega_M)] = j_{H,*}(e^{\mu+\rho_H}).$$

Applying  $i_*$  to both sides and using Theorem 3.1.2 yields

$$i_*([V_H(\mu, \sigma + \omega_M)]) = i_*j_{H,*}(e^{\mu+\rho_H}) = j_{G,*}(e^{\mu+\rho_H}).$$

The result now follows from (3.2) plus the observation that, if  $w \in W_G$  maps the strictly  $H$ -dominant character  $\lambda = \mu + \rho_H$  to a strictly  $G$ -dominant element  $w(\lambda)$ , then  $w \in W^H$ . QED

**3.3.2. Example.** The character  $\rho_M$  is a highest weight of the spinor module  $S$  (see (2.5)) and hence is  $H$ -dominant. Therefore  $i_*([V_H(\rho_M, \omega_M)]) = 1$  by Theorem 3.3.1.

In the **Spin** case Kostant [19], [20] has established a refinement of Theorem 3.3.1, which amounts to a vanishing theorem for the kernel or the cokernel of the Dirac operator (taken with respect to a very special connection), and which yields a  $\mathbb{Z}/2\mathbb{Z}$ -graded generalization of the classical Borel-Weil-Bott theorem. Kostant's theorem can be extended to the general maximal rank case by means of our techniques, but we will not pursue that avenue here.

**3.4. Duality and induction.** There is a natural pairing between twisted representation modules defined by means of twisted **Spin**<sup>c</sup>-induction. As before, we let  $\tau$  be a central extension of  $H$  by  $\mathbf{U}(1)$ . Define

$$\mathcal{P}: R(H, \tau) \times R(H, \omega_M - \tau) \longrightarrow R(G)$$

by  $\mathcal{P}(a_1, a_2) = i_*(a_1 a_2)$  for  $a_1 \in R(H, \tau)$  and  $a_2 \in R(H, \omega_M - \tau)$ . It follows from the  $R(G)$ -linearity of the induction map  $i_*$  that  $\mathcal{P}$  is  $R(G)$ -bilinear. Recall that  $A^\vee$  denotes the dual of an  $R(G)$ -module  $A$ . We call the pairing *nonsingular* if the two  $R(G)$ -linear maps

$$\mathcal{P}^\sharp: R(H, \tau) \longrightarrow R(H, \omega_M - \tau)^\vee, \quad R(H, \omega_M - \tau) \longrightarrow R(H, \tau)^\vee$$

induced by  $\mathcal{P}$  are isomorphisms. The map  $\mathcal{P}^\sharp$  fits into a diagram

$$\begin{array}{ccc} R(T) & \xrightarrow{\mathcal{P}_\delta^\sharp} & R(T)^\vee \\ \iota \updownarrow \pi & & \iota^\vee \updownarrow \pi^\vee \\ R(H, \tau) & \xrightarrow{\mathcal{P}^\sharp} & R(H, \omega_M - \tau)^\vee. \end{array} \quad (3.3)$$

Here  $\mathcal{P}_\delta$  is the  $R(G)$ -bilinear pairing on  $R(T)$  defined by holomorphic induction (cf. Example 3.2.3),

$$\mathcal{P}_\delta(b_1, b_2) = j_{G, \delta}(b_1 b_2) = j_{G, *}(e^{\rho_G} b_1 b_2)$$

for  $b_1, b_2 \in R(T)$ . Choose a character  $\mu$  of level 1 of the extended torus  $T^{(\tau)}$ . Then  $e^\mu$  is a free generator of the  $R(T)$ -module  $R(T, \tau)$  (see Example 1.2.3), and we define  $\iota$ ,  $\pi$ ,  $\iota^\vee$ , and  $\pi^\vee$  to be the compositions of the following  $R(H)$ -linear maps:

$$\iota: R(H, \tau) \xrightarrow{j_H^*} R(T, \tau) \xrightarrow{e^{-\mu}} R(T), \quad (3.4)$$

$$\pi: R(T) \xrightarrow{e^{\rho_{H^+} + \mu}} R(T, \omega_{H/T} + \tau) \xrightarrow{j_{H, *}} R(H, \tau), \quad (3.5)$$

$$\iota^\vee: R(H, \omega_M - \tau)^\vee \xrightarrow{t(j_{H, *})} R(T, \omega_{G/T} - \tau)^\vee \xrightarrow{t(e^{\rho_G - \mu})} R(T)^\vee,$$

$$\pi^\vee: R(T)^\vee \xrightarrow{t(e^{-\rho_{M^+} + \mu})} R(T, \omega_M - \tau)^\vee \xrightarrow{t(j_H^*)} R(H, \omega_M - \tau)^\vee.$$

Here  $e^\lambda$  denotes “multiplication by  $e^\lambda$ ” and  $t(e^\lambda)$  the transpose operator, etc. We show next that the diagram (3.3) commutes in two different ways and that the  $\iota$ 's are sections of the  $\pi$ 's.

**3.4.1. Lemma.** *We have*

$$\iota^\vee \circ \mathcal{P}^\sharp = \mathcal{P}_\delta^\sharp \circ \iota, \quad \pi^\vee \circ \mathcal{P}^\sharp = \mathcal{P}^\sharp \circ \pi, \quad \pi \circ \iota = \text{id}, \quad \pi^\vee \circ \iota^\vee = \text{id}.$$

*In particular the  $R(H)$ -module  $R(H, \tau)$  is isomorphic to a direct summand of  $R(T)$ .*

*Proof.* Let  $a \in R(H, \tau)$  and  $b \in R(T)$ . Then

$$\iota^\vee(\mathcal{P}^\sharp(a))(b) = \mathcal{P}^\sharp(a)(j_{H,*}(e^{\rho_G - \mu}b)) = i_*(aj_{H,*}(e^{\rho_G - \mu}b))$$

and

$$\mathcal{P}_\delta^\sharp(\iota(a))(b) = j_{G,*}(e^{\rho_G}e^{-\mu}j_H^*(a)b) = i_*(j_{H,*}(e^{\rho_G - \mu}j_H^*(a)b)) = i_*(aj_{H,*}(e^{\rho_G - \mu}b)),$$

where we used functoriality of induction,  $j_{G,*} = i_* \circ j_{H,*}$  (Theorem 3.1.2). Therefore  $\iota^\vee \circ \mathcal{P}^\sharp = \mathcal{P}_\delta^\sharp \circ \iota$ . The proof that  $\pi^\vee \circ \mathcal{P}^\sharp = \mathcal{P}^\sharp \circ \pi$  is similar. Let  $a \in R(H)$ . Then

$$\pi(\iota(a)) = j_{H,*}(e^{\rho_H + \mu}e^{-\mu}j_H^*(a)) = j_{H,*}(e^{\rho_H}j_H^*(a)) = j_{H,*}(e^{\rho_H})a = a,$$

because  $j_{H,*}(e^{\rho_H}) = 1$  (e.g. by Theorem 3.3.1). The proof that  $\pi^\vee \circ \iota^\vee = \text{id}$  is similar. QED

We can now establish a basic structure result for the twisted modules  $R(H, \tau)$ , which generalizes theorems of Pittie [27] and Steinberg [36] and Kazhdan and Lusztig [18].

**3.4.2. Theorem.** *Assume that  $\pi_1(G)$  is torsion-free. Then the pairing  $\mathcal{P}$  is nonsingular and  $R(H, \tau)$  is a free  $R(G)$ -module of rank  $|W^H|$ .*

*Proof.* Kazhdan and Lusztig [18, Proposition 1.6] showed that the map  $\mathcal{P}_\delta^\sharp$  is an isomorphism. Together with Lemma 3.4.1 this proves that  $\mathcal{P}^\sharp$  is also an isomorphism. Upon replacing  $\tau$  with  $\omega_M - \tau$  we see that the map

$$R(H, \omega_M - \tau) \longrightarrow R(H, \tau)^\vee$$

induced by  $\mathcal{P}$  is likewise an isomorphism. Therefore  $\mathcal{P}$  is nonsingular. The Pittie-Steinberg theorem [27], [36] says that  $R(H)$  is a free  $R(G)$ -module of rank  $|W^H|$ . By Lemma 3.4.1,  $R(H, \tau)$  is isomorphic to a direct summand of  $R(T)$ , so applying Pittie-Steinberg to  $H = T$  we find that  $R(H, \tau)$  is a projective  $R(G)$ -module. Moreover, since  $R(T)$  is finitely generated and  $R(G)$  is a Noetherian ring,  $R(H, \tau)$  is finitely generated. Steinberg proved also that  $R(G)$  is the tensor product of a polynomial algebra and a Laurent polynomial algebra over  $\mathbf{Z}$ . The assertion that  $R(H, \tau)$  is free now follows from the theorem of Quillen [28] and Suslin [37], which states that finitely generated projective modules over such rings are free. (Quillen and Suslin proved their result for polynomial algebras over a principal ideal domain and Suslin remarked that his proof works just as well for mixed polynomial and Laurent polynomial algebras. See also [22, Chapter V].) Lemma A.2 (see Appendix A) states that  $\text{rank}_{R(H)}(R(H, \tau)) = 1$ . By the Pittie-Steinberg theorem we conclude that

$$\text{rank}_{R(G)}(R(H, \tau)) = \text{rank}_{R(G)}(R(H)) \cdot \text{rank}_{R(H)}(R(H, \tau))$$

is equal to  $|W^H|$ . QED

In general it is not true that the twisted module  $R(H, \tau)$  is free over  $R(H)$ . (See Example A.3.) Steinberg [36] has constructed an explicit basis of the  $R(G)$ -module  $R(H)$ . We do not know if there is a similar construction for a basis of  $R(H, \tau)$ .

The next result, which is in essence [7, Theorem III], highlights the contrast between maximal rank subgroups and other subgroups of  $G$  (cf. Theorem 1.3.4(ii)): in the maximal rank case *every* twisted induction map (at least for  $\sigma = 0$ ) arises from twisted equivariant elliptic operators.

**3.4.3. Theorem.** *Assume that  $\pi_1(G)$  is torsion-free. Then the group of twisted induction maps  $R(H, \tau)^\vee$  is a free  $R(G)$ -module of rank  $|W^H|$ . Every twisted induction map  $f \in R(H, \tau)^\vee$  is of the form  $f = \mathcal{P}^\sharp(a)$  for a unique  $a \in R(H, \omega_M - \tau)$ . Choose  $H^{(\omega_M - \tau)}$ -modules  $V_0$  and  $V_1$  of level 1 such that  $a = [V_0] - [V_1]$ . Let  $D_0$  and  $D_1$  be the twisted Dirac operators defined by the equivariant Clifford modules  $S \otimes V_0$ , resp.  $S \otimes V_1$ . Then  $f = i_{D_0} - i_{D_1}$ .*

*Proof.* The first two assertions follow immediately from Theorem 3.4.2. We have  $i_{D_0}(b) = i_*([V_0]b)$  and  $i_{D_1}(b) = i_*([V_1]b)$  for all  $b \in R(H, \tau)$  by Example 2.1.3. Hence  $f(b) = i_*(ab) = i_{D_0}(b) - i_{D_1}(b)$  for all  $b$ . QED

#### 4. APPLICATIONS TO K-THEORY

The results of the previous sections lead to some direct consequences in equivariant K-theory. We use the same notation as in §§ 1–3. Recall that  $G$  denotes a compact connected Lie group,  $H$  a closed connected subgroup of maximal rank,  $T$  a common maximal torus of  $H$  and  $G$ , and  $\tau$  a central extension of  $H$  by  $\mathbf{U}(1)$ . In addition we denote by  $X$  a compact topological  $G$ -space, by  $K_G^*(X)$  its  $G$ -equivariant K-ring, and by  $K_H^*(X, \tau)$  its  $\tau$ -twisted  $H$ -equivariant K-group.

**4.1. The Künneth theorem and duality.** We establish an equivariant Künneth formula, Proposition 4.1.1, which is a minor variation on results of Hodgkin [17], Snaith [35], McLeod [26], and Rosenberg and Schochet [29], and which allows us to extend the results of §2 to equivariant K-theory. This formula implies that the  $G$ -equivariant K-group  $K_G^*(X)$  is a direct summand of the (untwisted)  $H$ -equivariant K-group  $K_H^*(X)$ , and in Theorem 4.1.3 we identify this direct summand in terms of linear equations. Theorem 4.1.4 is a duality theorem, which generalizes the nonsingular pairing  $\mathcal{P}$  from representation rings to equivariant K-theory.

First we extend the definition of twisted induction maps to K-theory. The action map

$$A: G \times X \rightarrow X$$

defined by  $A(g, x) = gx$  is  $G$ -equivariant with respect to the left multiplication action on  $G$  and  $H$ -invariant with respect to the action  $h \cdot (g, x) = (gh^{-1}, hx)$ . In the sense of [5] the triple  $\mathbf{X} = (G \times X, X, A)$  is a  $G \times H$ -equivariant family of smooth manifolds over  $X$  with fibre  $G$  and projection  $A$ . The  $H$ -action on  $\mathbf{X}$  is free and the quotient family  $\tilde{\mathbf{X}} = G \times^H X$  is a (trivial)  $G$ -equivariant family of smooth manifolds over  $X$  with fibre  $M = G/H$  and projection map  $\tilde{A}: \tilde{\mathbf{X}} \rightarrow X$  given by  $\tilde{A}([g, x]) = gx$ .

Now let  $U^0$  and  $U^1$  be  $H^{(\tau)}$ -modules of level 1 and let

$$D: \Gamma(G, G \times U^0) \longrightarrow \Gamma(G, G \times U^1)$$

be a twisted equivariant elliptic differential operator on  $M$ . Let  $V$  be an  $H^\tau$ -equivariant vector bundle of level  $-1$  over  $X$ . Then  $E^p = U^p \otimes V$  is an  $H$ -equivariant vector bundle over  $X$ , where  $p = 0, 1$ . The pullback  $\mathbf{E}^p = A^*E^p$  is a  $G \times H$ -equivariant vector bundle over the family  $\mathbf{X}$ . We may assume without loss of generality that  $\mathbf{E}^p$  is smooth along the fibres of  $\mathbf{X}$ , because any Gauss map for  $\mathbf{E}^p$  can be approximated by one which is smooth along the fibres. As in [1, §2] we extend  $D$  to a  $G \times H$ -equivariant family of differential operators

$$\mathbf{D}_{V,x}: \Gamma(A^{-1}(x), \mathbf{E}^0|A^{-1}(x)) \longrightarrow \Gamma(A^{-1}(x), \mathbf{E}^1|A^{-1}(x))$$

whose symbol is equal to

$$\text{symbol}(\mathbf{D}_V) = \text{symbol}(D) \otimes \text{id}_V.$$

It follows that each operator  $\mathbf{D}_{V,x}$  in the family is transversely elliptic with respect to the  $H$ -action, so  $\mathbf{D}_V$  descends to a  $G$ -equivariant elliptic family

$$\tilde{\mathbf{D}}_{V,x}: \Gamma(\tilde{A}^{-1}(x), \tilde{\mathbf{E}}^0|\tilde{A}^{-1}(x)) \longrightarrow \Gamma(\tilde{A}^{-1}(x), \tilde{\mathbf{E}}^1|\tilde{A}^{-1}(x))$$

over  $\tilde{\mathbf{X}}$  with coefficients in the quotient bundle  $\tilde{\mathbf{E}}^p = G \times^H E^p$ . We define the induction map

$$i_D: K_H^*(X, -\tau) \longrightarrow K_G^*(X)$$

by  $i_D([V]) = \text{index}(\tilde{\mathbf{D}}_V)$ . More generally, if  $\sigma$  is a central extension of  $G$  by  $\mathbf{U}(1)$ , this construction gives an induction map

$$i_D: K_H^*(X, \sigma - \tau) \longrightarrow K_G^*(X, \sigma),$$

which is linear over the ring  $K_G^*(X)$ . Taking  $D$  to be the twisted  $\mathbf{Spin}^c$  Dirac operator  $\mathcal{D}$  we get the looked-for map

$$i_* = i_{\mathcal{D}}: K_H^*(X, \sigma + \omega_M) \longrightarrow K_G^*(X, \sigma). \quad (4.1)$$

Replacing  $G$  with  $H$  and  $H$  with  $T$  gives the map  $j_{H,*}: K_T^*(X, \tau + \omega_{H/T}) \rightarrow K_H^*(X, \tau)$ , which occurs in the proof of the following Künneth formula.

**4.1.1. Proposition.** *Assume that  $\pi_1(G)$  is torsion-free. Then the map*

$$\phi: R(H, \tau) \otimes_{R(G)} K_G^*(X) \longrightarrow K_H^*(X, \tau)$$

*defined by  $\phi(u \otimes b) = u \cdot i^*(b)$  is an isomorphism of  $\mathbf{Z}/2\mathbf{Z}$ -graded  $R(H)$ -modules. Hence  $K_H^*(X, \tau)$  is a free module over  $K_G^*(X)$  of rank  $|W^H|$ .*

*Proof.* Evidently  $\phi$  is an  $R(H)$ -linear morphism of degree 0. To prove that it is bijective we consider a diagram similar to (3.3),

$$\begin{array}{ccc} R(T) \otimes_{R(G)} K_G^*(X) & \xrightarrow{\psi} & K_T^*(X) \\ \iota \otimes \text{id} \updownarrow \pi \otimes \text{id} & & \updownarrow \iota \pi \\ R(H, \tau) \otimes_{R(G)} K_G^*(X) & \xrightarrow{\phi} & K_H^*(X, \tau). \end{array}$$

Here  $\psi$  is defined by  $\psi(v \otimes b) = v \cdot j_G^*(b)$  and the maps  $\iota$  and  $\pi$  on the left are defined in (3.4) and (3.5). Analogously the maps  $\iota$  and  $\pi$  on the right are defined



to be the compositions

$$\begin{aligned} \iota: K_H^*(X, \tau) &\xrightarrow{j_H^*} K_T^*(X, \tau) \xrightarrow{e^{-\mu}} K_T^*(X), \\ \pi: K_T(X) &\xrightarrow{e^{\rho_H + \mu}} K_T^*(X, \omega_{H/T} + \tau) \xrightarrow{j_{H,*}} K_H^*(X, \tau). \end{aligned}$$

As in Lemma 3.4.1 we have the properties

$$\iota \circ \phi = \psi \circ (\iota \otimes \text{id}), \quad \pi \circ \psi = \phi \circ (\pi \otimes \text{id}),$$

which follow from the  $K_H^*(X)$ -linearity of the restriction map  $j_H^*$  and the induction map  $j_{H,*}$ ; and the property  $\pi \circ \iota = \text{id}$ , which follows from  $j_{H,*}(e^{\rho_H}) = 1$ . The map  $\psi$  is an isomorphism by the usual equivariant Künneth theorem of [29]. Hence  $\phi$  is also an isomorphism. The second statement of the proposition now follows from Theorem 3.4.2. QED

**4.1.2. Example.** Assume that  $\pi_1(G)$  is torsion-free. Taking  $X = G$  gives an isomorphism

$$K^*(G/H, \tau) \cong R(H, \tau) \otimes_{R(G)} \mathbf{Z},$$

and taking  $X = G/H$  gives an isomorphism

$$K_T^*(G/H, \tau) \cong R(H, \tau) \otimes_{R(G)} R(H).$$

The equivariant Künneth formula expresses  $H$ -equivariant K-theory in terms of  $G$ -equivariant K-theory, but it also enables us to do the reverse. Let  $\mathcal{E} = \text{End}_{R(G)}(R(H))$  be the ring of  $R(G)$ -linear endomorphisms of  $R(H)$  and let  $I(\mathcal{E})$  be the left ideal

$$I(\mathcal{E}) = \{ \Delta \in \mathcal{E} \mid \Delta(1) = 0 \}.$$

For any left  $\mathcal{E}$ -module  $A$  let  $A^{I(\mathcal{E})}$  denote the subgroup of elements annihilated by  $I(\mathcal{E})$ . The Künneth isomorphism, Proposition 4.1.1, shows that  $K_H^*(X)$  is in a natural way a left  $\mathcal{E}$ -module if  $\pi_1(G)$  is torsion-free. A version of the next theorem for the maximal torus  $H = T$  was proved in [15, Theorem 4.6]. (However, the version for the maximal torus is stronger: it is less restrictive in that it does not require the assumption on  $\pi_1(G)$ , and it is more explicit in that there is a description of the left ideal  $I(\mathcal{E})$  in terms of divided difference operators.)

**4.1.3. Theorem.** *Assume that  $\pi_1(G)$  is torsion-free. Then the map  $i^*: K_G^*(X) \rightarrow K_H^*(X)$  is an isomorphism onto  $K_H^*(X)^{I(\mathcal{E})}$ .*

*Proof.* The group  $R(H)$  is an  $\mathcal{E}$ - $R(G)$ -bimodule. By the Pittie-Steinberg theorem it is free of finite rank  $|W^H|$  as a  $R(G)$ -module. Hence, by the Morita equivalence theorem, the functor  $\mathfrak{G}: B \mapsto R(H) \otimes_{R(G)} B$  is an equivalence from the category of left  $R(G)$ -modules to the category of left  $\mathcal{E}$ -modules. The inverse of  $\mathfrak{G}$  is the functor  $\mathfrak{F}: A \mapsto \text{Hom}_{\mathcal{E}}(R(H), A)$ , which is isomorphic to the functor  $\mathfrak{J}: A \mapsto A^{I(\mathcal{E})}$ . (Cf. [15] for the case  $H = T$ .) In particular we can take  $A = K_H^*(X)$  and  $B = K_G^*(X)$ . Then  $A \cong \mathfrak{G}(B)$  by Proposition 4.1.1, so  $B \cong \mathfrak{J}(A) = K_H^*(X)^{I(\mathcal{E})}$ . QED

The pairing  $\mathcal{P}$  defined in §3.4 generalizes to a bi-additive pairing

$$\mathcal{P}_X: K_H^*(X, \tau) \times K_H^*(X, \omega_M - \tau) \longrightarrow K_G^*(X)$$

defined by  $\mathcal{P}_X(a_1, a_2) = i_*(a_1 a_2)$  for  $a_1 \in K_H^*(X, \tau)$  and  $a_2 \in K_H^*(X, \omega_M - \tau)$ . It follows from the naturality of  $i_*$  that

$$\mathcal{P}_X(f^*(a_1), f^*(a_2)) = f^* \mathcal{P}_Y(a_1, a_2) \quad (4.2)$$

for  $a_1 \in K_H^*(Y, \tau)$  and  $a_2 \in K_H^*(Y, \omega_M - \tau)$ , where  $f: X \rightarrow Y$  is any  $G$ -equivariant continuous map.

**4.1.4. Theorem.** *Assume that  $\pi_1(G)$  is torsion-free. Then the pairing  $\mathcal{P}_X$  is nonsingular. Hence*

$$K_H^*(X, \tau) \cong \text{Hom}_{K_G^*(X)}(K_H^*(X, \omega_M - \tau), K_G^*(X))$$

as  $\mathbf{Z}/2\mathbf{Z}$ -graded left  $K_G^*(X)$ -modules.

*Proof.* This follows from the nonsingularity of the pairing  $\mathcal{P}$  (Theorem 3.4.2), the Künneth theorem (Proposition 4.1.1) and the naturality property (4.2). (See [15, Proposition 5.1] for the case  $H = T$ .) QED

**4.1.5. Example.** Assume that  $\pi_1(G)$  is torsion-free. Let  $\tau = 0$ . Theorem 4.1.4 contains as special cases various forms of Poincaré duality in K-theory. For instance, taking  $X = G$  gives a nonsingular pairing

$$K^*(G/H) \times K^*(G/H, \omega_{G/H}) \longrightarrow \mathbf{Z},$$

and taking  $X = G/H$  gives a nonsingular pairing

$$K_H^*(G/H) \times K_H^*(G/H, \omega_{G/H}) \longrightarrow R(H).$$

If  $G/H$  has an invariant  $\mathbf{Spin}^c$ -structure, then  $\omega_{G/H} = 0$ , so we have a nonsingular  $\mathbf{Z}$ -valued pairing on  $K^*(G/H)$  and a nonsingular  $R(H)$ -valued pairing on  $K_H^*(G/H)$ .

**4.2. GKRS multiplets.** In this section we show how each  $T$ -equivariant K-class on a  $G$ -space  $X$  gives birth to a litter of  $T$ -equivariant classes parametrized by  $W^H$ , which we call a *Gross-Kostant-Ramond-Sternberg multiplet*. This is an extension of the notion of a multiplet introduced in [13] from a one-point space to an arbitrary  $G$ -space  $X$ . Generalizing a result of [13], we show that the alternating sum of a multiplet, when mapped to ordinary K-theory, vanishes.

We start by rewriting (2.7) in the “opposite” way. Define  $J_M^{\text{op}} \in \mathbf{Z}[W_G]$  by

$$J_M^{\text{op}} = \sum_{w \in W^H} \det(w) w^{-1}.$$

By decomposing  $w \in W$  as  $w = w''(w')^{-1}$  with  $w' \in W^H$  and  $w'' \in W_H$  we obtain

$$J_G = J_H J_M^{\text{op}}. \quad (4.3)$$

Let us write  $\partial_G = j_G^* \circ j_{G,*}$ . This is an  $R(G)$ -linear operator from  $R(T, \omega_{G/T})$  to  $R(T)$ , and the Weyl character formula for  $G$  states that  $\partial_G(a) = J_G(a)/\mathbf{d}_G$  for all  $a \in R(T, \omega_{G/T})$ . Similarly, for the group  $H$  we have  $\partial_H(b) = J_H(b)/\mathbf{d}_H$  for all  $b \in R(T, \omega_{H/T})$ . Combining this with (4.3) and substituting Lemma 2.2.1(iv) yields

$$\partial_G(a) = \frac{J_G(a)}{\mathbf{d}_G} = \frac{1}{j_H^*(\mathbf{e}(\mathcal{D}_M)^*)} \frac{J_H(J_M^{\text{op}}(a))}{\mathbf{d}_H} = \frac{1}{j_H^*(\mathbf{e}(\mathcal{D}_M)^*)} \partial_H(J_M^{\text{op}}(a))$$

for all  $a \in R(T, \omega_{G/T})$ . This amounts to an identity of  $R(G)$ -linear operators, namely

$$j_H^*(\mathbf{e}(\mathcal{D}_M)^*)\partial_G = \partial_H \circ J_M^{\text{op}}, \quad (4.4)$$

where the first operator on the left is multiplication by  $j_H^*(\mathbf{e}(\mathcal{D}_M)^*)$ . Except for the twists by the various orientation systems, which makes it work at the group level as opposed to the Lie algebra level, this identity is formula (1) in [13].

Because the induction maps  $j_{G,*}$  and  $j_{H,*}$  are defined in K-theory (see (4.1)), the operators  $\partial_G$  and  $\partial_H$  make sense in K-theory. The operator  $\partial_G$  maps  $K_T^*(X, \omega_{G/T})$  to  $K_T^*(X)$  and  $\partial_H$  maps  $K_T^*(X, \omega_{G/T})$  to  $K_T^*(X, \omega_M)$ .

$$K_T^*(X, \omega_{G/T} - \omega_{H/T}) = K_T^*(X, \omega_M).$$

The next lemma means that we can substitute in the identity (4.4) any class  $a \in K_T^*(X, \omega_{G/T})$ .

**4.2.1. Lemma.** *The following diagram commutes:*

$$\begin{array}{ccc} K_T^*(X, \omega_{G/T}) & \xrightarrow{\partial_G} & K_T^*(X) \\ J_M^{\text{op}} \downarrow & & \downarrow j_H^*(\mathbf{e}(\mathcal{D}_M)^*) \\ K_T^*(X, \omega_{G/T}) & \xrightarrow{\partial_H} & K_T^*(X, \omega_M). \end{array}$$

*Proof.* First assume that  $\pi_1(G)$  is torsion-free. Then the result follows by combining (4.4) with the Künneth formula (Proposition 4.1.1) and the  $K_G^*(X)$ -linearity of  $\partial_G$  and  $\partial_H$ . If  $\pi_1(G)$  is not torsion-free, we choose a covering  $\phi: \tilde{G} \rightarrow G$  of  $G$  by a compact connected  $\tilde{G}$  such that  $\pi_1(\tilde{G})$  is torsion-free, and we let  $\tilde{T}$  be the maximal torus  $\phi^{-1}(T)$  of  $\tilde{G}$ . Any extension  $T^{(\tau)}$  of  $T$  induces an extension  $\tilde{T}^{(\tau)}$  of  $\tilde{T}$ . It follows from [35, Lemma 2.4] that the pullback map

$$\phi^*: K_T^*(X, \tau) \longrightarrow K_{\tilde{T}}^*(X, \tau)$$

is injective. Since the kernel of  $\phi$  is a central subgroup of  $\tilde{G}$  and a  $W$ -invariant subgroup of  $\tilde{T}$ , the pullback map is natural with respect to each of the maps occurring in the identity (4.4):  $\phi^* \circ \partial_G = \partial_{\tilde{G}} \circ \phi^*$ , etc. Therefore the commutativity of the diagram for the group  $G$  follows from the commutativity for  $\tilde{G}$ . QED

For  $w \in W^H$  and  $a \in K_T^*(X, \omega_{G/T})$  put

$$a_w = \partial_H(w^{-1}(a)) \in K_T^*(X, \omega_M).$$

We call the  $W^H$ -tuple  $(a_w)_{w \in W^H}$  the *multiplet* generated by  $a$ . Let  $f: 1 \rightarrow T$  be the trivial homomorphism, which induces the forgetful map

$$f^*: K_T^*(X, \omega_M) \longrightarrow K^*(X, \omega_M).$$

**4.2.2. Theorem.** *Suppose that  $H \neq G$ . Let  $(a_w)_{w \in W^H}$  be a multiplet in  $K_T^*(X, \omega_M)$ . Then  $\sum_{w \in W^H} \det(w) f^*(a_w) = 0$ .*

*Proof.* Since  $H \neq G$ , we have  $\mathcal{R}_M \neq \emptyset$  and hence

$$f^* j_H^*(\mathbf{e}(\mathcal{D}_M)^*) = \prod_{\alpha \in \mathcal{R}_M^+} f^*(e^{-\alpha/2} - e^{\alpha/2}) = \prod_{\alpha \in \mathcal{R}_M^+} (1 - 1) = 0$$

by Lemma 2.2.1(iv). Therefore

$$\sum_{w \in W^H} \det(w) f^*(a_w) = f^* \partial_H J_M^{\text{op}}(a) = f^*(j_H^*(\mathbf{e}(\mathcal{D}_M)^*) \partial_G(a)) = 0$$

by Lemma 4.2.1.

QED

This result reduces to that of [13] by pulling back a multiplet on  $X$  to a point, i.e. by applying the augmentation map  $K_T^*(X, \omega_M) \rightarrow R(T, \omega_M)$ . It must be said that this K-theory version of the multiplet theorem is much weaker than the original version and the later version of [19]. For instance, there is no telling whether the elements of a multiplet are distinct or even nonzero. (However, suppose that  $\pi_1(G)$  is torsion-free and that  $a$  is of the form  $a = e^\lambda j_G^*(b)$ , where  $\lambda$  is a strictly  $G$ -dominant character and  $b \in K_G^*(X)$  is nonzero. Then  $a_w = \partial_H(w^{-1}(e^\lambda)) j_G^*(b)$ . The elements  $\partial_H(w^{-1}(e^\lambda))$  are restrictions to  $T$  of irreducible  $H$ -modules with distinct highest weights, and so by the Künneth formula the multiplet elements  $a_w$  are distinct and nonzero.)

#### APPENDIX A. CENTRAL EXTENSIONS

In this appendix we gather a few elementary facts regarding central extensions. The notation is as stated at the beginning of § 1. In addition  $\check{G}$  denotes a *connected* central extension of  $G$  by a *compact* abelian Lie group  $C$ ,

$$1 \longrightarrow C \longrightarrow \check{G} \longrightarrow G \longrightarrow 1.$$

A complex  $\check{G}$ -module  $V$  has *central character*  $\chi \in \mathcal{X}(C)$  if the subgroup  $C$  acts on  $V$  by  $c \cdot v = \chi(c)v$ . Let  $\mathfrak{Rep}^\chi(\check{G})$  be the category of finite-dimensional complex  $\check{G}$ -modules of central character  $\chi$ . We call the Grothendieck group  $R^\chi(\check{G})$  of  $\mathfrak{Rep}^\chi(\check{G})$  the  $\chi$ -*twisted representation module* of  $G$ . The category  $\mathfrak{Rep}^0(\check{G})$  (where  $\chi = 0$  is the trivial character) is equivalent to  $\mathfrak{Rep}(G)$ , so the groups  $R^0(\check{G})$  and  $R(G)$  are isomorphic. The tensor product functor

$$\mathfrak{Rep}^{\chi_1}(\check{G}) \times \mathfrak{Rep}^{\chi_2}(\check{G}) \longrightarrow \mathfrak{Rep}^{\chi_1 + \chi_2}(\check{G})$$

induces a bi-additive map

$$R^{\chi_1}(\check{G}) \times R^{\chi_2}(\check{G}) \longrightarrow R^{\chi_1 + \chi_2}(\check{G})$$

In particular  $R^\chi(\check{G})$  is an  $R(G)$ -module for all  $\chi$ .

- A.1. Lemma.** (i) *The  $R(G)$ -module  $R(\check{G})$  is the direct sum of the submodules  $R^\chi(\check{G})$  over all  $\chi \in \mathcal{X}(C)$ . Each summand  $R^\chi(\check{G})$  is nonzero.*  
(ii) *Let  $\check{T} \subseteq \check{G}$  be the inverse image of  $T$ . Then  $\check{T}$  is a maximal torus of  $\check{G}$ ; for each character  $\chi$  of  $C$  the submodule  $R^\chi(\check{T})$  of  $R(\check{T})$  is preserved by the  $W_G$ -action; and the restriction homomorphism  $R^\chi(\check{G}) \rightarrow R^\chi(\check{T})$  is an isomorphism onto  $R^\chi(\check{T})^{W_G}$ .*

*Proof.* Every  $\check{G}$ -module  $V$  decomposes under the action of  $C$  into a direct sum  $\bigoplus_{\chi \in \mathcal{X}(C)} V^\chi$  of isotypical submodules  $V^\chi$ . This decomposition is functorial and defines an equivalence of categories

$$\mathfrak{Rep}(\check{G}) \xrightarrow{\sim} \bigoplus_{\chi \in \mathcal{X}(C)} \mathfrak{Rep}^\chi(\check{G}).$$

Passing to Grothendieck groups we obtain the direct sum decomposition in (i). Each of the submodules  $R^\chi(\check{G})$  is nonzero, because there exists an irreducible representation of  $\check{G}$  with central character  $\chi$ , for instance an appropriate subrepresentation of the formally induced representation  $\text{ind}_{\check{C}}^{\check{G}}(\mathbf{C}_\chi)$ . Since  $C$  is central in  $\check{G}$ , the group  $\check{T}$  is a maximal torus of  $\check{G}$ , and the homomorphism  $\check{G} \rightarrow G$  induces isomorphisms of root systems  $\mathcal{R}_{\check{G}} \cong \mathcal{R}_G$  and of Weyl groups  $W_{\check{G}} \cong W_G$ . The homomorphism  $\check{T} \rightarrow T$  is  $W_G$ -equivariant. The  $W_G$ -action on  $\check{T}$  fixes  $C$  and therefore the submodule  $R^\chi(\check{T})$  of  $R(\check{T})$  is  $W_G$ -stable for each  $\chi \in \mathcal{X}(C)$ . The isomorphism  $R^\chi(\check{G}) \cong R^\chi(\check{T})^{W_G}$  now follows from  $R(\check{G}) \cong R(\check{T})^{W_G}$ . QED

Let  $V$  be an irreducible  $\check{G}$ -module. By Schur's lemma, the central subgroup  $C$  acts on  $V$  by a character  $\chi_V$ . Define

$$c \cdot [V] = \chi_V(c)[V] \quad (\text{A.1})$$

for  $c \in C$ . By linear extension, this formula defines a  $C$ -action on the complexified representation ring  $R(\check{G})_{\mathbf{C}}$  by ring automorphisms. The ring of  $C$ -invariants is

$$(R(\check{G})_{\mathbf{C}})^C \cong R(G)_{\mathbf{C}}. \quad (\text{A.2})$$

Recall that the *rank* of a module  $A$  over a domain  $R$  is the dimension of the vector space  $A \otimes_R F$ , where  $F$  is the fraction field of  $R$ , and is denoted by  $\text{rank}_R(A)$ .

**A.2. Lemma.**  $\text{rank}_{R(G)}(R^\chi(\check{G})) = 1$  for every  $\chi \in \mathcal{X}(C)$ .

*Proof.* Assume first that  $C$  is finite. Since  $G$  and  $\check{G}$  are connected, the rings  $R(G)$  and  $R(\check{G})$  have no zero divisors, so we can form the fraction fields  $\mathbf{K}$  of  $R(G)_{\mathbf{C}}$  and  $\check{\mathbf{K}} = R(\check{G})_{\mathbf{C}} \otimes_{R(G)_{\mathbf{C}}} \mathbf{K}$  of  $R(\check{G})_{\mathbf{C}}$ . It follows from (A.2) that  $\check{\mathbf{K}}$  is a Galois extension of  $\mathbf{K}$  with Galois group  $C$ , which implies  $\dim_{\mathbf{K}}(\check{\mathbf{K}}) = |C|$ . For  $\chi \in \mathcal{X}(C)$  let

$$\check{\mathbf{K}}^\chi = R^\chi(\check{G})_{\mathbf{C}} \otimes_{R(G)_{\mathbf{C}}} \mathbf{K},$$

which is a  $\mathbf{K}$ -linear subspace of  $\check{\mathbf{K}}$ . It follows from Lemma A.1(i) that as a vector space over  $\mathbf{K}$

$$\check{\mathbf{K}} = \bigoplus_{\chi \in \mathcal{X}(C)} \check{\mathbf{K}}^\chi,$$

where each of the summands is nonzero. The number of summands is  $|C|$  because of the fact that  $\mathcal{X}(C) \cong C$ , and therefore  $\dim_{\mathbf{K}}(\check{\mathbf{K}}^\chi) = 1$  for all  $\chi$ . Hence

$$\text{rank}_{R(G)}(R^\chi(\check{G})) = \dim_{\mathbf{K}}(\check{\mathbf{K}}^\chi) = 1.$$

For general  $C$  we make the basic observation, which appears to go back to Shapiro [31], that the compact central extension  $\check{G}$  is the pushout of a *finite* central extension

$$1 \longrightarrow Z \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1.$$

One produces  $\tilde{G}$  by choosing a Lie algebra homomorphism  $\kappa: \mathfrak{g} \rightarrow \check{\mathfrak{g}}$  which splits the exact sequence

$$0 \rightarrow \mathfrak{c} \rightarrow \check{\mathfrak{g}} \rightarrow \mathfrak{g} \rightarrow 0.$$

One can choose  $\kappa$  to be defined over  $\mathbf{Q}$ ; this ensures that it exponentiates to a Lie group homomorphism  $\kappa: \tilde{G} \rightarrow \check{G}$ , where  $\tilde{G}$  is a finite connected covering group of  $G$ . Let  $Z$  be the kernel of the covering  $\tilde{G} \rightarrow G$ ; then  $\kappa(Z)$  is contained in  $C$ . Let

$\chi$  be a character of  $C$  and  $\kappa^*(\chi)$  its pullback to  $Z$ . Then the  $R(G)$ -module  $R^\chi(\check{G})$  is isomorphic to  $R^{\kappa^*(\chi)}(\check{G})$ , and we already know that the latter is of rank 1. QED

**A.3. Example.** Let  $\check{G}_1 = \check{G}_2 = \mathbf{SU}(2)$  and

$$\check{G} = \check{G}_1 \times \check{G}_2 \cong \mathbf{Spin}(4), \quad C = \{\pm(I, I)\}, \quad G = \check{G}/C \cong \mathbf{SO}(4).$$

Then  $R(\check{G}) \cong R(\check{G}_1) \otimes R(\check{G}_2) \cong \mathbf{Z}[x_1, x_2]$ . Identify  $\mathcal{X}(C)$  with  $\mathbf{Z}/2\mathbf{Z} = \{0, 1\}$ ; then  $C$  acts on the monomial  $x_1^{r_1} x_2^{r_2} \in R(\check{G})$  with weight  $(r_1 + r_2) \bmod 2$ . Hence  $R(\check{G})$  is the direct sum of the submodules

$$R(G) \cong R^0(\check{G}) = \bigoplus_{r_1+r_2 \equiv 0} \mathbf{Z} \cdot x_1^{r_1} x_2^{r_2}, \quad R^1(\check{G}) = \bigoplus_{r_1+r_2 \equiv 1} \mathbf{Z} \cdot x_1^{r_1} x_2^{r_2},$$

where the congruences are modulo 2. As a ring,

$$R(G) \cong \mathbf{Z}[y_1, y_2, y_3] / (y_1 y_2 - y_3^2),$$

where the inclusion  $R(G) \rightarrow R(\check{G})$  is given by

$$y_1 \mapsto x_1^2, \quad y_2 \mapsto x_2^2, \quad y_3 \mapsto x_1 x_2.$$

The twisted module  $R^1(\check{G})$  is generated by  $x_1$  and  $x_2$ , which are subject to the single relation  $y_3 x_1 - y_1 x_2 = 0$ . Over the quotient field of  $R(G)$  the two generators are multiples of each other,  $x_2 = (y_2 y_3^{-1}) x_1$  and  $x_1 = (y_1 y_3^{-1}) x_2$ , so  $R^1(\check{G})$  is of rank 1, as predicted by Lemma A.2. However,  $R^1(\check{G})$  is not generated by any single element and is therefore not free.

## APPENDIX B. SHIFTED ANTI-INVARIANTS

This appendix is devoted to the proof of the statement below (Proposition 2.2.5 in the main text). Here  $\omega_{G/T}$  is the orientation system of the flag variety  $G/T$  defined in (2.2),  $J_G$  is the antisymmetrizer (2.6), and  $\mathbf{d}_G$  is the Weyl denominator (2.8). Recall that  $A^{-W_G}$  denotes the set of anti-invariant elements of a  $W_G$ -module  $A$ . If  $G$  is semisimple and simply connected, then  $\rho_G \in \mathcal{X}(T)$  and therefore the  $R(T)$ -module  $R(T, \omega_{G/T})$  is  $W_G$ -equivariantly isomorphic to  $R(T)$  by Lemma 2.2.1(iii). The proposition is then a standard fact; see e.g. [8, § VI.3, Proposition 2]. We will deduce the general case from this special case.

- Proposition.** (i) *The  $W_G$ -action on  $R(T^{(\omega_{G/T})})$  preserves  $R(T, \omega_{G/T})$ .*  
(ii) *The set of anti-invariants  $R(T, \omega_{G/T})^{-W_G}$  is a free  $R(G)$ -module of rank 1 generated by  $\mathbf{d}_G$ .*  
(iii) *The elements  $J_G(e^\lambda)$ , with  $\lambda \in \rho_G + \mathcal{X}(T)$  strictly dominant, form a basis of the  $\mathbf{Z}$ -module  $R(T, \omega_{G/T})^{-W_G}$ .*

*Proof.* Put

$$W = W_G, \quad J = J_G, \quad \omega = \omega_{G/T}, \quad \mathbf{d} = \mathbf{d}_G, \quad \rho = \rho_G.$$

We identify  $R(T)$  with  $\mathbf{Z}[\mathcal{X}(T)]$  and  $R(T, \omega)$  with  $R(T) e^\rho = \mathbf{Z}[\rho + \mathcal{X}(T)]$  as in Remark 2.2.2. The fact that  $\rho - w(\rho)$  is in the root lattice for all  $w \in W$  implies that the  $W$ -action on  $\mathcal{X}(T)_{\mathbf{Q}}$  preserves the affine lattice  $\rho + \mathcal{X}(T)$ . This proves (i). Let  $\phi: \tilde{G} \rightarrow G$  be a compact connected covering group which is the product  $\tilde{G} = C \times \bar{G}$  of a torus  $C$  and a simply connected group  $\bar{G}$ . Let  $\tilde{T}$  be the maximal torus  $\phi^{-1}(T)$  of  $\tilde{G}$ , and identify  $\mathcal{X}(T)$  with its image  $\phi^*(\mathcal{X}(T))$ . Let  $\tilde{T}$  be the

maximal torus  $\bar{G} \cap \tilde{T}$  of  $\bar{G}$ . Since  $\bar{G}$  is simply connected,  $\rho \in \mathcal{X}(\tilde{T})$  and therefore  $\mathbf{d} \in R(\tilde{T})$ . We have

$$R(\tilde{T})^W = R(C \times \tilde{T})^W = (R(C) \otimes_{\mathbb{Z}} R(\tilde{T}))^W = R(C) \otimes_{\mathbb{Z}} R(\tilde{T})^W,$$

because  $W$  acts trivially on  $R(C)$  and  $R(C)$  is a free abelian group. Since  $\bar{G}$  is simply connected, it follows from [8, § VI.3, Proposition 2] that

$$R(\tilde{T})^{-W} = R(\tilde{T})^W \cdot \mathbf{d}.$$

Therefore

$$\begin{aligned} R(\tilde{T})^{-W} &= R(C \times \tilde{T})^{-W} = (R(C) \otimes_{\mathbb{Z}} R(\tilde{T}))^{-W} = R(C) \otimes_{\mathbb{Z}} R(\tilde{T})^{-W} \\ &= R(C) \otimes_{\mathbb{Z}} (R(\tilde{T})^W \cdot \mathbf{d}) = (R(C) \otimes_{\mathbb{Z}} R(\tilde{T})^W) \cdot \mathbf{d} = R(\tilde{T})^W \cdot \mathbf{d}. \end{aligned} \quad (\text{B.1})$$

Now let  $a \in R(T, \omega)^{-W} \subseteq R(\tilde{T})^{-W}$ . It follows from (B.1) that  $a = b\mathbf{d}$  for some  $b \in R(\tilde{T})^W$ . We need to argue that  $b \in R(T)$ . Let  $K \subseteq \tilde{T}$  be the kernel of the covering homomorphism  $\phi: \tilde{G} \rightarrow G$ . This group acts on the complexified representation ring  $R(\tilde{T})_{\mathbb{C}}$  as in (A.1), and the ring of  $K$ -invariants  $(R(\tilde{T})_{\mathbb{C}})^K$  is isomorphic to  $R(T)_{\mathbb{C}}$ . Since  $a$  and  $\mathbf{d}$  are in  $R(T, \omega) = R(T)e^{\theta}$ , we have  $k \cdot a = \rho(k)a$  and  $k \cdot \mathbf{d} = \rho(k)\mathbf{d}$  for all  $k \in K$  and hence

$$b\mathbf{d} = a = \rho(k)^{-1}k \cdot a = \rho(k)^{-1}k \cdot (b\mathbf{d}) = \rho(k)^{-1}(k \cdot b)(k \cdot \mathbf{d}) = (k \cdot b)\mathbf{d}.$$

It follows that  $k \cdot b = b$  for all  $k \in K$ , i.e.  $b \in R(T)$ . This proves (ii). (iii) is proved in exactly the same way as [8, § VI.3, Proposition 1]. QED

### APPENDIX C. HOMOGENEOUS $\mathbf{Spin}^c$ -STRUCTURES

In this appendix we review the classification of invariant  $\mathbf{Spin}^c$ -structures on equal-rank homogeneous spaces, which is surely well-known but for which we could not find a reference. (But see [16, § 2.6] and also Example C.5 below for remarks on the  $\mathbf{Spin}$  case.) See [25] for examples of maximal-rank homogeneous spaces that do not carry invariant  $\mathbf{Spin}^c$ -structures.

The notation is as explained at the beginning of §§ 1 and 2. In particular recall that  $\mathfrak{m} = T_1 M$  denotes the tangent space at the identity coset of the homogeneous space  $M = G/H$  and that  $\eta: H \rightarrow \mathbf{SO}(\mathfrak{m})$  denotes the tangent representation. We denote the set of equivalence classes of  $G$ -invariant  $\mathbf{Spin}^c$ -structures on  $M$  by  $\mathfrak{Spin}_G^c(M)$ .

**C.1. Definition.** The orthogonal representation  $\eta$  is *c-spinorial* if it lifts to a homomorphism  $H \rightarrow \mathbf{Spin}^c(\mathfrak{m})$ . The subgroup  $H$  is *c-spinorial* if  $\eta$  is c-spinorial.

Note that  $\eta$  is c-spinorial if and only if  $M$  possesses a  $G$ -invariant  $\mathbf{Spin}^c$ -structure, i.e. if and only if  $\mathfrak{Spin}_G^c(M)$  is nonempty. Moreover, liftings of  $\eta$  to  $\mathbf{Spin}^c(\mathfrak{m})$  correspond bijectively to elements of  $\mathfrak{Spin}_G^c(M)$ .

A lifting of  $\eta$  to  $\mathbf{Spin}^c(\mathfrak{m})$  determines a trivialization  $s$  of the central extension

$$\omega = \omega_M: 1 \longrightarrow \mathbf{U}(1) \longrightarrow H^{(\omega)} \xrightarrow[\phi]{s} H \longrightarrow 1$$

defined in (2.2). Conversely, given a section  $s$  of  $\phi$  we can compose it with the canonical homomorphism  $\hat{\eta}: H^{(\omega)} \rightarrow \mathbf{Spin}^c(\mathfrak{m})$  to obtain a lifting of  $\eta$ ,

$$H \xrightarrow{s} H^{(\omega)} \xrightarrow{\hat{\eta}} \mathbf{Spin}^c(\mathfrak{m}).$$

Thus we have a natural one-to-one correspondence between  $\mathfrak{Spin}_G^c(M)$  and the set of trivializations of the orientation system  $\omega$ .

The *determinant character* is the character  $\det: H^{(\omega)} \rightarrow \mathbf{U}(1)$  obtained by pulling back the determinant character of  $\mathbf{Spin}^c(\mathfrak{m})$ , which is defined by  $\det([z, a]) = z^2$ . The homomorphism

$$\psi = \det \times \phi: H^{(\omega)} \longrightarrow \mathbf{U}(1) \times H$$

is a double covering map. Let  $s$  be a section of  $\phi$  and let  $\gamma$  be the character  $\det \circ s$  of  $H$ . Then  $s$  is a lifting homomorphism of  $\gamma \times \text{id}$ , as in the commutative diagram

$$\begin{array}{ccc} & & H^{(\omega)} \\ & \nearrow s & \downarrow \psi \\ H & \xrightarrow{\gamma \times \text{id}} & \mathbf{U}(1) \times H, \end{array}$$

and so  $s = s_\gamma$  is uniquely determined by  $\gamma$ .

**C.2. Definition.** A character  $\gamma \in \mathcal{X}(H)$  is *c-spinorial* (relative to the orthogonal  $H$ -module  $\mathfrak{m}$ ) if the lifting  $s_\gamma$  of  $\gamma \times \text{id}$  exists. We denote the set of *c-spinorial* characters by  $\mathcal{X}(H)^c$ .

The conclusion is that we have natural bijections between the set  $\mathfrak{Spin}_G^c(M)$ , the set of trivializations of  $\omega$ , and the set  $\mathcal{X}(H)^c$ . This proves the first part of the following proposition. The map  $\nu: \mathcal{X}(H) \rightarrow \frac{1}{2}\mathcal{X}(T)$  in the second part is defined by

$$\nu(\gamma) = \frac{1}{2}J_H^*(\gamma) - \frac{1}{2} \sum_{\alpha \in \mathcal{R}_M^+} \alpha = \frac{1}{2}J_H^*(\gamma) - \rho_M.$$

**C.3. Proposition.** (i) For a *c-spinorial* character  $\gamma$ , define a  $G$ -invariant principal  $\mathbf{Spin}^c(\mathfrak{m})$ -bundle over  $M$  by

$$\mathbf{P}_\gamma = G \times^H \mathbf{Spin}^c(\mathfrak{m}),$$

where the right-hand side denotes the quotient of  $G \times \mathbf{Spin}^c(\mathfrak{m})$  by the  $H$ -action  $h \cdot (g, k) = (gh^{-1}, \hat{\eta}(s_\gamma(h))k)$ . The map

$$f: \mathcal{X}(H)^c \longrightarrow \mathfrak{Spin}_G^c(M)$$

which sends  $\gamma$  to the equivalence class of  $\mathbf{P}_\gamma$  is bijective. In particular  $\mathfrak{Spin}_G^c(M)$  is nonempty if and only if  $\mathcal{X}(H)^c$  is nonempty.

(ii) A character  $\gamma$  of  $H$  is *c-spinorial* if and only if  $\nu(\gamma) \in \mathcal{X}(T)$ . Hence  $\mathcal{X}(H)^c = \nu^{-1}(\mathcal{X}(T))$ .

*Proof.* It remains to prove (ii). By covering space theory and our standing assumption that  $H$  is connected, a lifting  $s$  of  $\gamma \times \text{id}$  exists if and only if it exists on the level of fundamental groups, as in the diagram

$$\begin{array}{ccc} & & \pi_1(H^{(\omega)}) \\ & \nearrow s_* & \downarrow \psi_* \\ \pi_1(H) & \xrightarrow{\gamma_* \times \text{id}} & \pi_1(\mathbf{U}(1)) \times \pi_1(H). \end{array} \tag{C.1}$$



Let  $\mathcal{Y}(T) = \text{Hom}(\mathbf{U}(1), T)$  denote the cocharacter group of  $T$ . The fundamental group  $\pi_1(H)$  is naturally isomorphic to  $\mathcal{Y}(T)/Q(\mathcal{R}_H^\vee)$ , where  $Q(\mathcal{R}_H^\vee)$  is the coroot lattice in  $\mathcal{Y}(T)$ . (See e.g. [8, §IX.4.6].) Since  $\psi$  is a covering map,  $\psi_*$  maps the coroot system of  $H^{(\omega)}$  bijectively to that of  $\mathbf{U}(1) \times H$ . Therefore the lifting problem (C.1) is equivalent to the lifting problem

$$\begin{array}{ccc} & \mathcal{Y}(T^{(\omega)}) & \\ & \swarrow s_* & \downarrow \psi_* \\ \mathcal{Y}(T) & \xrightarrow{\gamma_* \times \text{id}} & \mathcal{Y}(\mathbf{U}(1)) \oplus \mathcal{Y}(T). \end{array}$$

Since  $\mathcal{X}(T)$  and  $\mathcal{Y}(T)$  are dual abelian groups, this lifting problem is equivalent to the dual extension problem

$$\begin{array}{ccc} & \mathcal{X}(T^{(\omega)}) & \\ & \nwarrow s^* & \uparrow \psi^* = \det^* \times \phi^* \\ \mathcal{X}(T) & \xleftarrow{\gamma^* \oplus \text{id}} & \mathcal{X}(\mathbf{U}(1)) \oplus \mathcal{X}(T). \end{array} \quad (\text{C.2})$$

By Lemma 2.2.1(ii),  $\mathcal{X}(T^{(\omega)})$  is the direct sum of  $\phi^*(\mathcal{X}(T))$  and  $\mathbf{Z} \cdot (\varepsilon_0 - \rho_M)$ . Let  $\mathcal{X}$  be the image of  $\psi^*$ , which is a sublattice of  $\mathcal{X}(T^{(\omega)})$  of index 2. Since  $2\rho_M$  is in  $\mathcal{X}(T)$  and  $\psi^*$  maps the generator  $\varepsilon_0$  of  $\mathcal{X}(\mathbf{U}(1))$  to the determinant character  $\det = 2\varepsilon_0$ , the element  $\varepsilon_0 - \rho_M \in \mathcal{X}(T^{(\omega)})$  is a representative of the nontrivial coset in  $\mathcal{X}(T^{(\omega)})/\mathcal{X}$ . Because  $2(\varepsilon_0 - \rho_M)$  is equal to  $\psi^*(\varepsilon_0 - 2\rho_M)$ , the extension problem (C.2) is soluble if and only if

$$2\nu(\gamma) = j_H^*(\gamma) - 2\rho_M = \gamma^*(\varepsilon_0) - 2\rho_M \in \mathcal{X}(T)$$

is divisible by 2 in  $\mathcal{X}(T)$ . If this is the case, the extension  $s^* = s_\gamma^*$  is uniquely determined by the formula

$$s_\gamma^*(\varepsilon_0 - \rho_M) = \nu(\gamma) \in \mathcal{X}(T). \quad (\text{C.3})$$

This proves (ii). QED

We call  $\gamma \in \mathcal{X}(H)^c$  the *character* of the  $\mathbf{Spin}^c$ -structure  $\mathbf{P}_\gamma$ . Let  $\gamma \in \mathcal{X}(H)^c$  and  $\chi \in \mathcal{X}(H)$ . Then

$$\nu(\gamma + 2\chi) = \nu(\gamma) + j_H^*(\chi) \in \mathcal{X}(T),$$

so  $\gamma + 2\chi \in \mathcal{X}(H)^c$ . Thus we have an action of the abelian group  $\mathcal{X}(H)$  on the set of  $c$ -spinorial characters defined by  $\chi \cdot \gamma = \gamma + 2\chi$ . Let  $\mathbf{L}_\chi = G \times^H \mathbf{C}_\chi$  be the homogeneous complex line bundle on  $M$  defined by  $\chi \in \mathcal{X}(H)$ . Recall the natural isomorphisms

$$\mathcal{X}(H) \xrightarrow{\cong} \text{Pic}_G(M) \xrightarrow{\cong} H_G^2(M, \mathbf{Z}), \quad (\text{C.4})$$

where  $\text{Pic}_G(M)$  denotes the (topological) Picard group of isomorphism classes of homogeneous complex line bundles on  $M$ . The first map sends a character  $\chi$  to the class of the bundle  $\mathbf{L}_\chi$  and the second map sends the class of a bundle  $L$  to its

equivariant Chern class  $c_G^1(L)$ . (See e.g. [14, Theorem C.47].) A natural action of  $\text{Pic}_G(M)$  on  $\mathfrak{Spin}_G^c(M)$  is defined by

$$[L] \cdot [\mathbf{P}] = [\mathbf{U}(L) \times^{\mathbf{U}(1)_M} \mathbf{P}],$$

where  $\mathbf{U}(L)$  denotes the circle bundle associated to  $L$  and we identify the circle  $\mathbf{U}(1)$  with the kernel of  $\psi: \mathbf{Spin}^c(\mathfrak{m}) \rightarrow \mathbf{SO}(\mathfrak{m})$ . (The quotient in the right-hand side is taken in the category of manifolds over  $M$ , and  $\mathbf{U}(1)_M = M \times \mathbf{U}(1)$  denotes the constant groupoid over  $M$  with fibre  $\mathbf{U}(1)$ .)

**C.4. Proposition.** *The notation is as in Proposition C.3.*

- (i) *The bijection  $f: \mathcal{X}(H)^c \rightarrow \mathfrak{Spin}_G^c(M)$  is  $\mathcal{X}(H)$ -equivariant with respect to the group isomorphism (C.4).*
- (ii)  *$\mathfrak{Spin}_G^c(M)$  is a principal homogeneous space for the abelian group  $\mathcal{X}(H)$ . In particular  $\mathfrak{Spin}_G^c(M)$  consists of at most one element if  $H$  is semisimple.*
- (iii) *Let  $\gamma \in \mathcal{X}(H)^c$ . The determinant line bundle of  $\mathbf{P}_\gamma$  is  $\mathbf{L}_\gamma = G \times^H \mathbf{C}_\gamma$ . Its equivariant Chern class  $c_1(\mathbf{L}_\gamma) \in H_G^2(M, \mathbf{Z}) \cong \mathcal{X}(H)$  is equal to  $\gamma$ .*
- (iv) *Let  $\gamma \in \mathcal{X}(H)^c$  and let  $\delta_\gamma$  be the  $\mathbf{Spin}^c$  Dirac operator associated with  $\mathbf{P}_\gamma$ . The equivariant Euler class of  $\delta_\gamma$  is  $\mathbf{e}(\delta_\gamma) = e^{\gamma/2} \mathbf{e}(\mathcal{D}) \in R(H)$ , where  $\mathcal{D}$  is the twisted  $\mathbf{Spin}^c$  Dirac operator of  $M$ .*

*Proof.* Let  $\gamma \in \mathcal{X}(H)^c$  and  $\chi \in \mathcal{X}(H)$ . The fibre of  $\mathbf{U}(\mathbf{L}_\chi) \times^{\mathbf{U}(1)_M} \mathbf{P}_\gamma$  over the identity coset  $\bar{1} \in M$  is

$$\mathbf{U}(\mathbf{C}_\chi) \times^{\mathbf{U}(1)} \mathbf{Spin}^c(\mathfrak{m}),$$

where  $\mathbf{U}(\mathbf{C}_\chi)$  denotes the unit circle in  $\mathbf{C}_\chi$ . The map

$$\mathbf{U}(\mathbf{C}_\chi) \times^{\mathbf{U}(1)} \mathbf{Spin}^c(\mathfrak{m}) \longrightarrow \mathbf{Spin}^c(\mathfrak{m})$$

defined by  $(z, k) \mapsto zk$  is a diffeomorphism and is equivariant with respect to the  $H$ -actions defined on the left-hand side by

$$h \cdot [z, k] = [\chi(h)^{-1}z, \hat{\eta}(s_\gamma(h))k]$$

and on the right-hand side by  $h \cdot k = \hat{\eta}(s_{\gamma+2\chi}(h))k$ . It follows that  $[L_\chi] \cdot [\mathbf{P}_\gamma] = [\mathbf{P}_{\gamma+2\chi}]$ , which proves (i). It is easy to verify that the  $\mathcal{X}(H)$ -action on  $\mathcal{X}(H)^c$  is free and transitive. Thus (ii) follows from (i). The determinant line bundle of  $\mathbf{P}_\gamma$  is

$$\mathbf{L}_\gamma = \mathbf{P}_\gamma \times^{\mathbf{Spin}^c(\mathfrak{m})} \mathbf{C}_{\det} = (G \times^H \mathbf{Spin}^c(\mathfrak{m}))^{\mathbf{Spin}^c(\mathfrak{m})} \mathbf{C}_{\det} = G \times^H \mathbf{C}_\gamma.$$

The isomorphism  $\mathcal{X}(H) \rightarrow H_G^2(M, \mathbf{Z})$  defined in (C.4) maps  $\gamma$  to  $c_G^1(\mathbf{L}_\gamma)$ , which proves (iii). By (1.9) and (2.3), the Euler class of  $\delta_\gamma$  is

$$\mathbf{e}(\delta_\gamma) = (\hat{\eta} \circ s_\gamma)^*([S^0] - [S^1]) = s_\gamma^* \hat{\eta}^*([S^0] - [S^1]) = s_\gamma^*(\mathbf{e}(\mathcal{D})) \in R(H),$$

where  $S$  is the spinor module of  $\mathbf{Cl}(\mathfrak{m}_{\mathbf{C}})$ . Using Lemma 2.2.1(iv) and (C.3) we obtain

$$j_H^*(\mathbf{e}(\delta_\gamma)) = s_\gamma^* \left( e^{\varepsilon_0 - \rho_M} \prod_{\alpha \in \mathcal{R}_M^+} (1 - e^\alpha) \right) = e^{\nu(\gamma)} \prod_{\alpha \in \mathcal{R}_M^+} (1 - e^\alpha) = j_H^*(e^{\gamma/2} \mathbf{e}(\mathcal{D})),$$

which establishes (iv). QED

**C.5. Example (Spin-structures).** Let us call the orthogonal representation  $\eta: H \rightarrow \mathbf{SO}(\mathfrak{m})$ , or the subgroup  $H$ , *spinorial* if  $\eta$  lifts to a homomorphism  $\tilde{\eta}: H \rightarrow \mathbf{Spin}(\mathfrak{m})$ . Thus,  $H$  is spinorial if and only if  $M$  has a  $G$ -invariant **Spin**-structure, and such a structure is unique up to equivalence, because  $H$  is connected. A **Spin**-structure determines a **Spin**<sup>c</sup>-structure by means of the homomorphism  $\kappa \circ \tilde{\eta}: H \rightarrow \mathbf{Spin}^c(\mathfrak{m})$ , where  $\kappa: \mathbf{Spin}(\mathfrak{m}) \rightarrow \mathbf{Spin}^c(\mathfrak{m})$  is the inclusion map. The corresponding  $c$ -spinorial character of  $H$  is  $\gamma = \det \circ \kappa \circ \tilde{\eta} = 0$ , because **Spin**( $\mathfrak{m}$ ) is the kernel of the determinant character. Hence, by Proposition C.3(ii),  $M$  is  $G$ -invariantly **Spin** if and only if  $\rho_M \in \mathcal{X}(T)$ . The Euler class of the **Spin** Dirac operator is equal to that of the twisted **Spin**<sup>c</sup> Dirac operator  $\mathcal{D}$ .

**C.6. Remark.** If  $H$  is  $c$ -spinorial and semisimple, then a lifting  $H \rightarrow \mathbf{Spin}^c(\mathfrak{m})$  takes values in the commutator subgroup **Spin**( $\mathfrak{m}$ ), so  $H$  is spinorial.

**C.7. Example (almost complex structures).** Up to homotopy,  $G$ -invariant almost complex structures on  $M$  correspond bijectively to  $H$ -invariant orthogonal complex structures on  $\mathfrak{m}$ . Let  $J$  be such a structure; then the tangent representation  $\eta: H \rightarrow \mathbf{SO}(\mathfrak{m})$  takes values in  $\mathbf{U}(\mathfrak{m}, J)$ , the group of  $J$ -holomorphic orthogonal maps. We equip  $M$  with the **Spin**<sup>c</sup>-structure defined by composing this homomorphism with the canonical injection  $\iota$  (see e.g. [14, §D.3.1]),

$$H \xrightarrow{\eta} \mathbf{U}(\mathfrak{m}, J) \xhookrightarrow{\iota} \mathbf{Spin}^c(\mathfrak{m}).$$

The associated  $c$ -spinorial character  $\gamma$  is then  $\gamma = \det \circ \iota \circ \eta$ . Since  $\det \circ \iota$  is equal to the usual complex determinant character of  $\mathbf{U}(\mathfrak{m}, J)$ , we have  $j_H^*(\gamma) = \sum_{k=1}^l \beta_k$ , where the  $\beta_k$  are the weights of the  $T$ -action on  $\mathfrak{m}$  with respect to  $J$ . To compute this, let  $\mathcal{B}_M^+ = \{\alpha_1, \alpha_2, \dots, \alpha_l\}$  and let  $J_0$  be the  $T$ -invariant complex structure on  $\mathfrak{m}$  given by the decomposition  $\mathfrak{m} = \bigoplus_{k=1}^l \mathfrak{m}_k$ , where  $\mathfrak{m}_k \cong \mathfrak{g}_{\mathbb{C}}^{\alpha_k}$ . On  $\mathfrak{m}_k$  we have  $J = c_k J_0$ , where  $c_k = \pm 1$ . Hence  $\beta_k = c_k \alpha_k$  and  $j_H^*(\gamma) = \sum_{k=1}^l c_k \alpha_k$ .

**C.8. Example (complex structures).** As a special case of Example C.7, let us take  $H$  to be the centralizer of a subtorus of  $T$ . This is the case precisely when  $M$  has an integrable invariant almost complex structure, as one sees from the well-known identification  $M \cong G_{\mathbb{C}}/P$  ([8, §IX.4, Exercice 8]), where  $P$  is the parabolic subgroup of  $G_{\mathbb{C}}$  generated by  $H$  and the negative root spaces. The corresponding orthogonal almost complex structure  $J$  on  $\mathfrak{m}$  is positive with respect to the inner product on  $\mathfrak{m}$ , so  $c_k = 1$  for all  $k$  and hence

$$j_H^*(\gamma) = 2\rho_M, \quad \nu(\gamma) = 0, \quad j_H^*(\mathbf{e}(\delta_\gamma)) = \prod_{k=1}^l (1 - e^{\alpha_k}).$$

The Kähler structure on  $M$  and the associated **Spin**<sup>c</sup>-structure depend on the choice of the basis of  $\mathcal{B}_G$ .

## NOTATION INDEX

$\bar{1}$ , identity coset  $1H \in M$ , 3

$A \times_C B$ , fibred product, 3

$\alpha$ , root of  $G$ , 3

$\mathcal{B}_G$ , basis of  $\mathcal{B}_G$ , 3

$\mathbf{C}_\chi$ , one-dimensional  $G$ -module defined by character  $\chi \in \mathcal{X}(G)$ , 3

$\mathbf{Cl}(\mathfrak{m}_{\mathbb{C}})$ , Clifford algebra of  $\mathfrak{m}_{\mathbb{C}}$ , 9

$\mathbf{d}_G$ , Weyl denominator of  $G$ , 16

$\mathcal{D}$ , twisted **Spin**<sup>c</sup> Dirac operator, 11

$\delta$ , **Spin**<sup>c</sup> Dirac operator, 19

- $e^\chi$ , class of  $C_\chi$  in  $R(G)$ , 3  
 $e(D)$ , Euler class of operator  $D$ , 8  
 $\varepsilon_0$ , standard generator of  $\mathcal{R}(\mathbf{U}(1)) \cong \mathbf{Z}$ , 13  
 $\eta$ , tangent representation  $H \rightarrow \mathbf{GL}(\mathfrak{m})$ , 3  
 $G$ , compact connected Lie group, 3  
 $G \times^H V$ , homogeneous vector bundle, 3  
 $G^{(\sigma)}$ , central extension of  $G$  by  $\mathbf{U}(1)$ , 4  
 $\Gamma$ , smooth global sections functor, 3  
 $\gamma$ , c-spinorial character of  $H$ , 19  
 $\mathfrak{g}$ , Lie algebra of  $G$ , 3  
 $\mathfrak{g}_C^\alpha$ , root space of  $\mathfrak{g}_C$ , 10  
 $H$ , closed subgroup of  $G$ , from §1.5 onward  
 connected and containing  $T$ , 3  
 $H^{(\tau)}$ , central extension of  $H$  by  $\mathbf{U}(1)$ , 6  
 $i$ , inclusion  $H \rightarrow G$ , 3  
 $i_!$ , formal induction, 3  
 $i_*$ , twisted  $\mathbf{Spin}^c$ -induction, 12  
 $i_b$ ,  $\mathbf{Spin}^c$ -induction, 19  
 $i_D$ , induction defined by operator  $D$ , 7  
 $\text{ind}_H^G$ , formal induction, 3  
 $j_G$ , inclusion  $T \rightarrow G$ , 3  
 $j_H$ , inclusion  $T \rightarrow H$ , 9  
 $J_G$ , antisymmetrizer of  $W_G$ , 15  
 $J_M$ , relative antisymmetrizer, 16  
 $J_M^{\text{op}}$ , “opposite” of  $J_M$ , 26  
 $M$ , homogeneous space  $G/H$ , 3  
 $\mathfrak{m}$ , tangent space  $T_1 M$ , 3  
 $\mathfrak{m}_C$ , complexified module  $\mathbf{C} \otimes_{\mathbf{R}} \mathfrak{m}$ , 10  
 $\mathfrak{m}^\alpha$ , weight space, 10  
 $\omega_M$ , orientation system of  $M$ , 12  
 $\pi$ , projection  $T^*M \rightarrow M$ , 12  
 $R(G)$ , representation ring, 3  
 $R(G)^*$ , dual  $\mathbf{Z}$ -module of  $R(G)$ , 3  
 $R(G, \sigma)$ , twisted representation module, 4  
 $R(H)^\vee$ , dual  $R(G)$ -module of  $R(H)$ , 4  
 $\mathcal{R}_G$ , root system of  $G$ , 3  
 $\mathcal{R}_G^+$ , positive roots of  $G$ , 3  
 $\mathcal{R}_M$ , weights  $\mathcal{R}_G \setminus \mathcal{R}_H$  of  $\mathfrak{m}_C$ , 10  
 $\mathcal{R}_M^+$ , positive weights  $\mathcal{R}_G^+ \setminus \mathcal{R}_H^+$  of  $\mathfrak{m}_C$ , 10  
 $\rho_G$ , half-sum of positive roots of  $G$ , 3  
 $\rho_M$ , half-character  $\rho_G - \rho_H$ , 11  
 $S$ , spinor module  $S^0 \oplus S^1$  of  $\mathbf{Cl}(\mathfrak{m}_C)$ , 12  
 $\sigma$ , central extension of  $G$  by  $\mathbf{U}(1)$ , 4  
 $\mathbf{Spin}_G^c(M)$ , set of equivalence classes of invariant  $\mathbf{Spin}^c$ -structures on  $M$ , 31  
 $T$ , maximal torus of  $G$ , 3  
 $T^\times M$ , punctured cotangent bundle  $T^*M \setminus \zeta(M)$ , 8  
 $\tau$ , central extension of  $H$  by  $\mathbf{U}(1)$ , 6  
 $\text{th}(D)$ , Thom class of  $D$ , 8  
 $\mathbf{U}(1)$ , unit circle, 3  
 $V_G(\lambda, \sigma)$ , irreducible  $G^{(\sigma)}$ -module of level 1 with highest weight  $\lambda$ , 20  
 $W_G$ , Weyl group of  $G$ , 3  
 $W^H$ , shortest representatives for  $W_G/W_H$ , 10  
 $X$ , topological  $G$ -space, 23  
 $\mathcal{X}(G)$ , character group  $\text{Hom}(G, \mathbf{U}(1))$ , 3  
 $\mathcal{X}(H)^c$ , c-spinorial characters of  $H$ , 32  
 $\zeta$ , zero section  $M \rightarrow T^*M$ , 8

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